POINTS IN A FOLD

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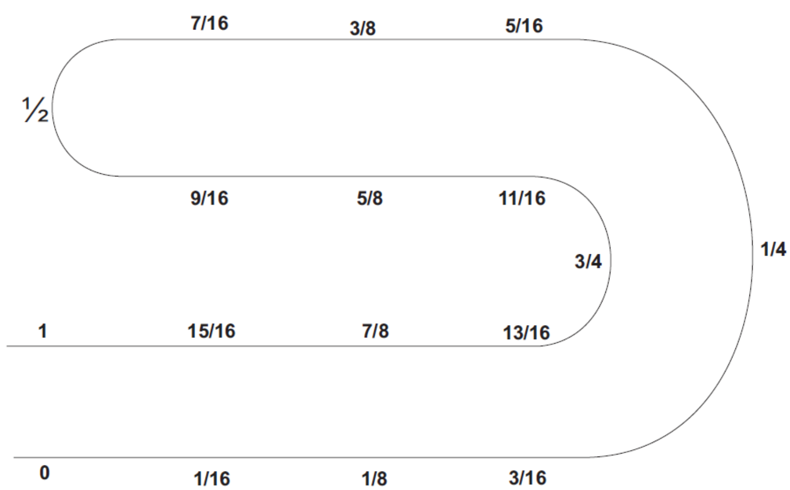
Abstract. When a page, represented by the interval $[0,1]$, is folded right over left n times, the right hand fold contans a sequence of points. This paper specifies these points and the order in which they appear in each fold. It also determines exactly where in the folded structure any point in $[0,1]$ appears and, given any point on the bottom line of the structure, which point lies at each level above it.

1. Introduction

When the line joining 0 to 1 (or a rectangular piece of paper with $[0,1]$ at its edge) is folded in half, right over left, the point $1/2$ is at the fold. When the line is then folded in half again, right over left, the points $1/4$ and $3/4$ are at this second fold, in that order, from the outside of the fold to its interior as shown in Figure 1.



Points in a fold after two folds

Note that we have ignored the fact that, strictly speaking, a line with no thickness, when halved and completely folded back upon itself, would have all points superimposed not juxtaposed at the fold.

Proceeding in this way, the sequence of points from the outside of the third fold to the inside becomes $\langle 1/8, 7/8, 5/8, 3/8\rangle$.

Definition 1. ***(nth fold sequence).*** *After we have folded the line n times, the sequence of points from the outside of the nth fold to the inside is designated as the* ***nth fold sequence,*** *S_n; and the jth term in S_n is designated as $S_{n,j}$.*

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Example 1. *We have*

$$S_2 = \langle S_{2,1}, S_{2,2} \rangle = \left\langle \frac{1}{4}, \frac{3}{4} \right\rangle .$$

The line after n folds has the following appearance:

- the nth fold, S_n, is on the right
- the ends of the line appear at the bottom on the left
- the first fold, S_1, then the second fold, S_2, all the way through to the $(n-1)$th fold, S_{n-1}, appear immediately above the ends of the line, in ascending order.

In this paper we will determine, for points found in folds in $[0,1]$:

- the jth term in S_n, that is, $S_{n,j}$, using only n and j. (Theorems 6 and 7)
- the relationship between points in the same fold. (Theorems 8 and 9)
- the exact position in the folded structure of any point of $[0,1]$.(Theorems 10 and 11)
- given a point on the bottom line of the structure, which point lies at any level above it (Theorem 5)

The line when unfolded shows a sequence of $\vee$ and $\wedge$ creases, which form the paperfolding sequence which has been studied in [1],[3],[4],[5] and [6].

2. Folds and Prefolds

We now show one of the properties of S_n:

Theorem 1. ***(Points of** S_n**).*** *For* $k = 0, 1, 2, \cdots, 2^{n-1} - 1$, *the points in* S_n *are of the form*

$$\frac{2k+1}{2^n}.$$

Proof. Each new fold places a crease midway between creases from earlier folds as well as a new crease midway between 0 and the first crease and another crease midway between 1 and the last crease. Thus the nth fold places a new crease at $1/2^n$ and further new creases thereafter at intervals of $1/2^{n-1}$ units. Thus for $k = 0, 1, 2, \cdots, 2^{n-1} - 1$, points in the nth fold occur at locations

$$\frac{1}{2^n} + \frac{k}{2^{n-1}} = \frac{2k+1}{2^n}.$$

□

We note that S_n is the sequence of middle points of the set of overlaid intervals, each of length $1/2^{n-1}$, formed after $n-1$ folds and listed from bottom to top. To help determine this order we introduce the concept of *prefolds.*

We begin with the definition of an n*th prefold sequence.*

Definition 2. ***(n*th prefold*).*** *After the line from* 0 *to* 1 *has been folded* n *times, where* $n \geq 0$, *let* j *be an integer such that* $0 < j \leq 2^n$ *and* a *be a real number such that* $0 < a < 1/2^n$. *Then we define*

1. $F_n(a,j)$ *as the* j*th term from the bottom, lying at, or directly above* a. *We call* $F_n(a,j)$ ***the*** j***th term of the*** n***th prefold sequence at*** a;
2. $F_n(a)$ *as the sequence of all terms in the* n***th prefold sequence at*** a. *That is,*

$$F_n(a) = \langle F_n(a,1), F_n(a,2), F_n(a,3), \cdots, F_n(a,2^n) \rangle \,;$$

(3) $\widetilde{F_n}(a)$ *as the* ***reverse*** n***th prefold sequence at*** a*, such that,*

$$\widetilde{F_n}(a) = \langle F_n(a, 2^n), \cdots, F_n(a,3), F_n(a,2), F_n(a,1)\rangle .$$

By Definition 2, $F_n(a,1) = a$ and

Theorem 2. *For* $n > 0$,

$$\begin{aligned} S_n &= F_{n-1}(\frac{1}{2^n}) \\ &= \left\langle F_{n-1}(\frac{1}{2^n},1), F_{n-1}(\frac{1}{2^n},2), F_{n-1}(\frac{1}{2^n},3), \cdots, F_{n-1}(\frac{1}{2^n},2^{n-1})\right\rangle . \end{aligned}$$

Example 2.

$$\begin{aligned} S_3 &= \left\langle F_2(\frac{1}{8},1), F_2(\frac{1}{8},2), F_2(\frac{1}{8},3), F_2(\frac{1}{8},4)\right\rangle \\ &= \left\langle \frac{1}{8}, \frac{7}{8}, \frac{5}{8}, \frac{3}{8}\right\rangle . \end{aligned}$$

We now show connections between prefolds.

Lemma 1. ***(Form of*** $F_n(a)$***)****. For* $0 < a < 1/2^n$,

$$F_n(a) = \left\langle F_{n-1}(a), \widetilde{F_{n-1}}(\frac{1}{2^{n-1}} - a)\right\rangle .$$

Proof. The nth fold occurs at $1/2^n$. It involves $F_{n-1}(\frac{1}{2^{n-1}} - a)$ for $0 < a < 1/2^n$ being rotated anti-clockwise around $1/2^n$ and then placed atop $F_{n-1}(a)$ to form $F_n(a)$, giving the result. □

By Lemma 1,

Theorem 3. ***(Recursive expression of prefolds)***

$$F_n(a,j) = \begin{cases} F_{n-1}(a,j), & \text{if } 1 \leq j \leq 2^{n-1} \\ F_{n-1}(\frac{1}{2^{n-1}} - a, 2^n - j + 1), & \text{if } 2^{n-1} < j \leq 2^n \end{cases}$$

Theorem 4. ***(Prefold expression of folds).***

$$\begin{aligned} S_n &= \left\langle F_{n-2}(\frac{1}{2^n}), \widetilde{F_{n-2}}(\frac{3}{2^n})\right\rangle \\ &= \Big\langle F_{n-2}(\frac{1}{2^n},1), F_{n-2}(\frac{1}{2^n},2), \cdots, F_{n-2}(\frac{1}{2^n},2^{n-2}), \\ &\qquad F_{n-2}(\frac{3}{2^n},2^{n-2}), F_{n-2}(\frac{3}{2^n},2^{n-2}-1), \cdots, F_{n-2}(\frac{3}{2^n},1)\Big\rangle . \end{aligned}$$

3. The Main Result

Theorem 5. *If* $s \leq 2^n$ *and*

(i) $s = j + 1$ *where* $j = \sum_{p=1}^{r} 2^{t_p} \leq 2^n$ *where* $r \geq 0$ *and if* $r > 0$, $t_1 > t_2 > \cdots > t_r > 0$, *then,*

$$F_n(a,s) = \sum_{p=1}^{r} \frac{1}{2^{t_p}} + a;$$

(ii) *$s = j$ with j as in (i), then,*

$$F_n(a,s) = \frac{1}{2^{t_r-1}} - \sum_{p=1}^{r-1} \frac{1}{2^{t_p}} - a.$$

Proof. By induction on n.

If $n = 1$ and $j = 0$ then $r = 0$ and $F_n(a, j+1) = a$ so we have (i).

If $n = 1$ and $j = 2$ then $r = t_r = 1$ and $F_n(a, j) = 1 - a$ so we have (ii).

Assume the result for n.

If $1 \le s \le 2^n$, by Theorem 3 (i) we have

$$F_{n+1}(a,s) = F_n(a,s).$$

So by the induction hypothesis, for $s = j$,we have (ii) with $n+1$ for n; and for $s = j+1 \le 2^n$,we have (i) with $n+1$ for n.

If $2^n < s \le 2^{n+1}$, we have $t_1 = n$ and by Theorem 3 (ii)

$$F_{n+1}(a,s) = F_n(\frac{1}{2^n} - a, 2^{n+1} - s + 1).$$

If $s = j+1$ and

$$2^{n+1} - (j+1) + 1 = \sum_{i=1}^{r-1} \left(\sum_{p=t_i-1}^{t_{i+1}+1} 2^p \right) + 2^{t_r}.$$

The double sum is the sum of the powers of 2 from 2^{t_1-1} to 2^{t_r+1} omitting $2^{t_2}, 2^{t_3}, \cdots, 2^{t_{r-1}}$.

By the induction hypothesis,

$$F_{n+1}(a, j+1) = \frac{1}{2^{t_r-1}} - \sum_{i=1}^{r-1} \left(\sum_{p=t_i-1}^{t_{i+1}+1} \frac{1}{2^p} \right) + a - \frac{1}{2^n},$$

where the double sum is the sum of the powers of $\frac{1}{2}$ from $\frac{1}{2^{t_1-1}}$ to $\frac{1}{2^{t_r+1}}$, omitting $\frac{1}{2^{t_2}}, \frac{1}{2^{t_3}}, \cdots, \frac{1}{2^{t_{r-1}}}$.

So as

$$\frac{1}{2^{t_r-1}} - \frac{1}{2^n} = \frac{1}{2^{t_r}} + \frac{1}{2^{t_r+1}} + \cdots + \frac{1}{2^{t_1-1}} + \frac{1}{2^{t_1}},$$

$F_{n+1}(a,s)$ is as in (i).

If $s = j$,

$$2^{n+1} - j + 1 = \sum_{i=1}^{r-1} \left(\sum_{p=t_i-1}^{t_{i+1}+1} 2^p \right) + 2^{t_r} + 1.$$

By the induction hypothesis,

$$F_{n+1}(a,j) = \sum_{i=1}^{r-1} \left(\sum_{p=t_i-1}^{t_{i+1}+1} \frac{1}{2^p} \right) + \frac{1}{2^{t_r}} + \frac{1}{2^n} - a,$$

where the double sums are as in the $s = j+1$ case.

So as

$$\frac{1}{2^{t_r}} + \frac{1}{2^n} = \frac{1}{2^{t_r-1}} - \left(\frac{1}{2^{t_r+1}} + \cdots + \frac{1}{2^{t_1-1}} + \frac{1}{2^{t_1}} \right),$$

$F_{n+1}(a,s)$ is as in (ii). □

Theorem 6. *If $s \le 2^n$ and*

(i) $s = j + 1$ *where* $j = \sum_{p=1}^{r} 2^{t_p} \leq 2^n$ *where* $r \geq 0$ *and if* $r > 0$, $t_1 > t_2 > \cdots > t_r > 0$, *then,*

$$S_{n,s} = \sum_{p=1}^{r} \frac{1}{2^{t_p}} + \frac{1}{2^n};$$

(ii) $s = j$ *with* j *as in (i), then*

$$S_{n,s} = \frac{1}{2^{t_r - 1}} - \sum_{p=1}^{r-1} \frac{1}{2^{t_p}} - \frac{1}{2^n}.$$

Proof. By Theorem 5 with $a = \frac{1}{2^n}$. □

Example 3. *Since*

$$11 = 2^3 + 2 + 1,$$

so $r = 2, t_1 = 3$ *and* $t_2 = 1$ *and by Theorem 6 (i),*

$$\begin{aligned} S_{6,11} &= \frac{1}{2^3} + \frac{1}{2} + \frac{1}{2^6} \\ &= \frac{41}{64}. \end{aligned}$$

Similarly, since

$$26 = 2^4 + 2^3 + 2,$$

so $r = 3, t_1 = 4$ *and* $t_2 = 3$ *and by Theorem 6 (ii),*

$$\begin{aligned} S_{6,26} &= \frac{3}{2} - \left(\frac{1}{2^4} + \frac{1}{2^3} + \frac{1}{2} + \frac{1}{2^6} \right) \\ &= \frac{51}{64}. \end{aligned}$$

Here are some special cases of Theorem 6.

Theorem 7. *Let* $p \in \mathbb{N}$. *Then*

$$\begin{array}{ll} i)\ \text{If } n \geq 1, 0 < p \leq n, & S_{n,2^p - 1} = 1 - \frac{1}{2^{p-1}} + \frac{1}{2^n} \\ ii)\ \text{If } n > 1, 0 < p < n, & S_{n,2^p} = \frac{1}{2^{p-1}} - \frac{1}{2^n} \\ iii)\ \text{If } n > 2, 0 < p < n, & S_{n,2^p + 1} = \frac{1}{2^p} + \frac{1}{2^n} \\ iv)\ \text{If } n > 2, 0 < p < n, & S_{n,2^p + 2} = 1 - \frac{1}{2^p} - \frac{1}{2^n} \end{array}$$

We can relate consecutive elements of S_n.

Theorem 8. *If* $j = \sum_{p=1}^{r} 2^{t_p} \leq 2^{n-1}, r > 0$ *and* $t_1 > t_2 > \cdots > t_r > 0$, *then*

$$\begin{array}{ll} i) & S_{n,j+1} + S_{n,j} = \frac{3}{2^{t_p}} \\ ii) & S_{n,j-1} + S_{n,j} = 1 \end{array}$$

Proof. *i)* By Theorem 6.

ii) As $j - 1 = \sum_{p=1}^{r-1} 2^{t_p} + 2^{t_r - 1} + 2^{t_r - 2} + \cdots + 2 + 1$, by Theorem 6 (i),

$$\begin{aligned} S_{n,j-1} &= \sum_{p=1}^{r-1} \frac{1}{2^{t_p}} + \sum_{i=1}^{t_r - 1} \frac{1}{2^i} + \frac{1}{2^n} \\ &= \sum_{p=1}^{r-1} \frac{1}{2^{t_p}} + 1 - \frac{1}{2^{t_r}} + \frac{1}{2^n}. \end{aligned}$$

So as,

$$S_{n,j} = \frac{1}{2^{t_r}} - \sum_{p=1}^{r-1} \frac{1}{2^{t_p}} - \frac{1}{2^n}$$

we have the result. □

Theorem 9. *For $n \geq 2$,*

$$S_{n,j} = S_{n,2^{n-1}-j+1} + \frac{(-1)^j}{2^{n-1}}$$

Proof. i) For j even, if $j = \sum_{p=1}^{r} 2^{t_p} \leq 2^{n-1}$ where $r > 0$ and $t_1 > t_2 > \cdots > t_r > 0$, then

$$2^{n-1} - j + 1 = \sum_{i=t_1+1}^{n-2} 2^i + \sum_{i=1}^{r-1} \left(\sum_{p=t_{i+1}+1}^{t_i-1} 2^p \right) + 2^{t_r} + 1.$$

So by Theorem 6,

$$S_{n,2^{n-1}-j+1} = \sum_{i=t_1+1}^{n-2} \frac{1}{2^i} + \sum_{i=1}^{r-1} \left(\sum_{p=t_{i+1}+1}^{t_i-1} \frac{1}{2^p} \right) + \frac{1}{2^{t_r}} + \frac{1}{2^n}$$

and

$$S_{n,j} = \frac{1}{2^{t_r-1}} - \sum_{p=1}^{r-1} \frac{1}{2^{t_p}} - \frac{1}{2^n}.$$

So

$$\begin{aligned} S_{n,2^{n-1}-j+1} - S_{n,j} &= \sum_{i=t_1+1}^{n-2} \frac{1}{2^i} + \sum_{i=t_1}^{t_r+1} \frac{1}{2^i} - \frac{1}{2^{t_r}} + \frac{1}{2^{n-1}} \\ &= -\frac{1}{2^{n-1}}. \end{aligned}$$

ii) For j odd, let $j = \sum_{p=1}^{r} 2^{t_p} + 1$ with $r > 0$ and $t_1 > t_2 > \cdots > t_r > 0$.Then,

$$2^{n-1} - j + 1 = \sum_{i=t_1+1}^{n-2} 2^i + \sum_{i=1}^{r-1} \left(\sum_{p=t_{i+1}+1}^{t_i-1} 2^p \right) + 2^{t_r}.$$

So by Theorem 6,

$$S_{n,2^{n-1}-j+1} = \frac{1}{2^{t_r-1}} - \sum_{i=t_1+1}^{n-2} \frac{1}{2^i} + \sum_{i=1}^{r-1} \left(\sum_{p=t_{i+1}+1}^{t_i-1} \frac{1}{2^p} \right) - \frac{1}{2^n}$$

and

$$S_{n,j} = \sum_{p=1}^{r} \frac{1}{2^{t_p}} + \frac{1}{2^n}.$$

So

$$\begin{aligned} S_{n,2^{n-1}-j+1} - S_{n,j} &= \frac{1}{2^{t_r-1}} - \sum_{i=t_1+1}^{n-2} \frac{1}{2^i} - \sum_{i=t_1}^{t_r+1} \frac{1}{2^i} - \frac{1}{2^{n-1}} \\ &= \frac{1}{2^{n-1}}. \end{aligned}$$

□

The next two theorems allow us to determine exactly where in the n times folded structure any point b of $[0,1]$ appears.

Theorem 10. *If $0 \le b \le 1$, $b \ne \frac{2k+1}{2^{n+1}}$, then b lies on level j of the n folded structure at, or directly above, a, where $0 \le a < \frac{1}{2^n}$, where*

(i) *If $\lfloor 2^n b \rfloor$ is even, that is, $\lfloor 2^n b \rfloor = \sum_{p=1}^{s} 2^{u_p}$ where $s = 0$, or $s = 1$ and $u_1 = n$, or $n > u_1 > u_2 > \cdots > u_s > 0$, then*

$$a = b - \frac{\lfloor 2^n b \rfloor}{2^n} \text{ and}$$
$$j = 1 + \sum_{p=1}^{s} 2^{n-u_s-p+1}.$$

(ii) *If $\lfloor 2^n b \rfloor$ is odd, that is, for some $m, 0 \le m < n, \lfloor 2^n b \rfloor = 2^{n-m} - 1 - \sum_{p=1}^{v} 2^{w_p}$ where $v = 0$ or $v > 0$ and $n - m - 1 > w_1 > w_2 > \cdots > w_s > 0$, then,*

$$a = \frac{\lfloor 2^n b \rfloor}{2^n} + \frac{1}{2^n} - b \text{ and}$$
$$j = \sum_{p=1}^{v} 2^{n-w_v-p+1} + 2^{m+1}.$$

Proof. a and j are such that $0 \le a < \frac{1}{2^n}$, $1 \le j \le 2^n$ and $b = F_n(a,j)$.

(i) If $j = \sum_{p=1}^{r} 2^{t_p} + 1$, where if $r > 0, n > t_1 > t_2 > \cdots > t_r > 0$, by Theorem 5 (i),

$$b = \sum_{p=1}^{r} \frac{1}{2^{t_p}} + a.$$

So $\lfloor 2^n b \rfloor = \sum_{p=1}^{r} 2^{n-t_p}$, and if $\lfloor 2^n b \rfloor = \sum_{p=1}^{s} 2^{u_p}$, $r = s$ and for $1 \le p \le s, t_p = n - u_{s-p+1}$. So

$$j = 1 + \sum_{p=1}^{s} 2^{n-u_{s-p+1}}.$$

As $t_1 < n, j$ can only be odd if $u_s > 0$, that is, $\lfloor 2^n b \rfloor$ is even.

(ii) If $j = \sum_{p=1}^{r} 2^{t_p}$, where $r > 0$ and $t_1 > t_2 > \cdots > t_r > 0$, by Theorem 5 (ii),

$$b = \frac{1}{2^{t_r - 1}} - \sum_{p=1}^{r-1} \frac{1}{2^{t_p}} - a$$

So

$$\lfloor 2^n b \rfloor = 2^{n-t_r+1} - 1 - \sum_{p=1}^{r-1} 2^{n-t_p}.$$

So if j is even, $\lfloor 2^n b \rfloor$ must be odd. If for some $m, 0 \le m < n$,

$$\lfloor 2^n b \rfloor = 2^{n-m} - 1 - \sum_{p=1}^{v} 2^{w_p}$$

with $v = 0$ or $v > 0$ and $n - m - 1 > w_1 > w_2 > \cdots > w_p > 0$, we have

$$2^{n-m} - \sum_{p=1}^{v} 2^{w_p} = 2^{n-t_r+1} - \sum_{p=1}^{r-1} 2^{n-t_p}.$$

As $n-m > w_1+1$ and $n-t_r+1 > n-t_{r-1}+1$, we have $t_r = m+1, r = v+1$ and for $0 \le p \le r-1, t_p = n - w_{v-p+1}$. So

$$j = \sum_{p=1}^{v} 2^{n-w_{v-p+1}} + 2^{m+1}.$$

□

Example 4. *For $n = 5$ and*

$$\begin{aligned} b &= \frac{9}{32} + \frac{\pi}{180} \\ &= \frac{2^5 - 2^4 - 2^2 - 2 - 1}{2^5} + \frac{\pi}{180}. \end{aligned}$$

then $\lfloor 2^5 b \rfloor = 2^5 - 2^4 - 2^2 - 2 - 1 = 9$. So by Theorem 10 (ii) with $n = 5, m = 0, v = 3, w_1 = 4, w_2 = 2$ and $w_3 = 1$,

$$\begin{aligned} a &= \frac{5}{16} - \frac{9}{32} - \frac{\pi}{180} \\ &= \frac{1}{32} - \frac{\pi}{180} \text{ and} \\ j &= 2^4 + 2^3 + 2 \\ &= 26. \end{aligned}$$

For $n = 5$ and $b = \frac{5}{16} + \frac{\pi}{180}$, $\lfloor 2^5 b \rfloor = 2^3 + 2 = 10$. So by Theorem 10 (i) with $s = 2, u_1 = 3,$ and $u_2 = 1$,

$$\begin{aligned} a &= \frac{5}{16} + \frac{\pi}{180} - \frac{5}{16} \\ &= \frac{\pi}{180} \text{ and} \\ j &= 1 + 2^2 + 2^4 \\ &= 21. \end{aligned}$$

Theorem 11. *If $S_{n,j} = \frac{2k+1}{2^n}$ and*

(i) *$k = \sum_{p=1}^{s} 2^{u_p}$ where $n - 1 > u_1 > u_2 > \cdots > u_s > 0$ or $s = 1$ and $u_1 = n - 1$, then*

$$j = 1 + \sum_{p=1}^{s} 2^{n-1-u_{s-p+1}}.$$

(ii) *$k = 2^{n-1-m} - 1 - \sum_{p=1}^{v} 2^{w_p}$ where $v = 0$ or $v > 0$ and $n - m - 2 > w_1 > w_2 > \cdots > w_v > 0$, then*

$$j = 2^{m+1} + \sum_{p=1}^{v} 2^{n-1-w_{v-p+1}}.$$

Proof. If $S_{n,j} = \frac{2k+1}{2^n}$, then $F_{n-1}(\frac{1}{2^n}, j) = \frac{2k+1}{2^n}$ and $\lfloor 2^{n-1} \frac{2k+1}{2^n} \rfloor = k$, so we use Theorem 10 with $\frac{2k+1}{2^n}$ for b and $n - 1$ for n.

(i) If $k = \sum_{p=1}^{s} 2^{u_p}$ where $s = 1$ and $u_1 = n-1$ or $n-1 > u_1 > u_2 > \cdots > u_s > 0$, then

$$j = 1 + \sum_{p=1}^{s} 2^{n-1-u_{s-p+1}}.$$

(ii) If $k = 2^{n-1-m} - 1 - \sum_{p=1}^{v} 2^{w_p}$ where $v = 0$ or $v > 0$ and $n-m-2 > w_1 > w_2 > \cdots > w_v > 0$, then

$$j = 2^{m+1} + \sum_{p=1}^{v} 2^{n-1-w_{v-p+1}}.$$

□

Example 5. *For* $n = 5, k = 9, m = 0, v = 2, w_1 = 2, w_2 = 1,$ *we have* $S_{5,14} = \frac{19}{32}$.
For $n = 5, k = 14, u_1 = 3, u_2 = 2, u_3 = 1, s = 3$ *we have* $S_{5,15} = \frac{29}{32}$.

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