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January 2014

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Keywords

continued, fraction, brocot, stern

Publication Details

Bates, B. (2014). The Stern-Brocot continued fraction. Integers: electronic journal of combinatorial number theory, 14 A39-1-A39-23.

This journal article is available at Research Online: https://ro.uow.edu.au/eispapers/3529

#A39 INTEGERS 14 (2014)



THE STERN-BROCOT CONTINUED FRACTION

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Received: 10/18/12, Revised: 3/16/14, Accepted: 5/17/14, Published: 8/11/14

Abstract

We discover a continued fraction whose successive approximants generate the Stern–Brocot sequence and levels of the Stern–Brocot tree. We also discover continued fractions whose approximants generate every term in diagonals and branches of the Stern–Brocot tree.

1. Introduction and Preliminaries

The Stern–Brocot tree has received much attention recently due to its deep connections with physical chemistry [7]. Also recently, the application of continued fractions to the Stern–Brocot tree has greatly assisted in the understanding of the tree and the Stern–Brocot sequence to which it is related. For example, through the use of continued fractions we can now:

- describe the location of any term in the Stern–Brocot tree or its cousin, the Calkin–Wilf tree [3] and [2],
- describe the term that is found at any specific location in the Stern–Brocot tree or the Calkin–Wilf tree [2],
- provide a simple method for evaluating terms in the Hyperbinary sequence (a sequence related to the Calkin–Wilf tree) thereby answering a challenge raised in *Quantum* in September 1997 [2],
- translate terms from the Stern–Brocot tree to vertices in the Calkin–Wilf tree, and vice versa [2],
- show that the iterated Gauss map and the left half of the Stern–Brocot tree are analogues of each other [1],
- describe diagonals and branches within the Stern–Brocot tree [3] and [2], and
- generate results for the child's addition of continued fractions [5].

An excellent overview of current research into the Stern–Brocot tree is available at [9]. In this paper we go further by showing that there exists:

- a continued fraction (Theorem 12), which we style *the Stern–Brocot continued fraction*, whose successive approximants generate the Stern–Brocot sequence and the Stern–Brocot tree,
- an interleaving representation of the Stern–Brocot continued fraction (Theorem 21),
- a mirroring representation of the Stern–Brocot continued fraction (Theorem 25),
- a set of continued fractions for left and right diagonals in the Stern–Brocot tree (Theorem 30), and
- a set of continued fractions whose successive approximants generate branches (Theorem 37) and offset branches (Theorem 47) in the Stern–Brocot tree.

We state some important definitions and results. Proofs of results can be found in the references cited.

Definition 1. (**Stern–Brocot Sequence**). With $s_{0,1} = 0$ and $s_{0,2} = 1$, we define for $n \geq 0$,

$$S_n = \langle s_{n,1}, s_{n,2}, \ldots, s_{n,2^n+1} \rangle$$

as the sequence for which, for $k \geq 1, n > 0$,

$$\begin{aligned} s_{n,2k-1} &= s_{n-1,k} \text{ and} \\ s_{n,2k} &= s_{n-1,k} + s_{n-1,k+1} \end{aligned}$$

Similarly, with $q_{0,1} = 1$, $q_{0,2} = 0$, we define Q_n. Then the sequence defined by

$$H_n = \langle h_{n,1}, h_{n,2}, \ldots, h_{n,2^n+1} \rangle \text{ where } h_{n,i} = \frac{s_{n,i}}{q_{n,i}}$$

is called the *Stern–Brocot Sequence* of order n. It represents the sequence containing both the first n generations of mediants based on H_0, and the terms of H_0 itself.

Definition 2. (Parents). We call $h_{n-1,k}$ and $h_{n-1,k+1}$, the left and right *parents*, respectively, of $h_{n,2k}$.

Definition 3. (Levels of the Stern–Brocot tree). Let H_0 be level 0 of the Stern–Brocot tree. For $n > 0$, level n of the Stern–Brocot tree is defined as $\operatorname{med} H_{n-1}$ where

$$\begin{aligned} \operatorname{med} H_{n-1} &= \left\langle \left(h_{n-1,1} \oplus h_{n-1,2}\right), \left(h_{n-1,2} \oplus h_{n-1,3}\right), \ldots, \left(h_{n-1,2^{n-1}} \oplus h_{n-1,2^{n-1}+1}\right) \right\rangle, \\ &= \langle h_{n,2},\ h_{n,4},\ h_{n,6}, \ldots, h_{n,2^n} \rangle \end{aligned}$$

and $\oplus$ is the child's addition operator whereby

$$\frac{r}{m} \oplus \frac{c}{d} = \frac{r+c}{m+d}.$$

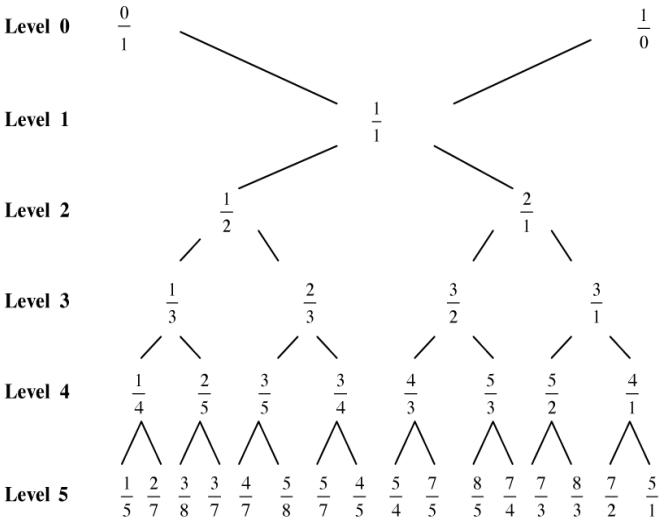


Figure 1: Levels 0 to 5 of the Stern-Brocot tree

Figure 1 shows levels 0 to 5 of the Stern–Brocot tree.

The following theorem is found in [3] and is adapted from [8].

Theorem 4. *For* $0 < i \leq 2^n$,

$$s_{n,i+1}q_{n,i} - s_{n,i}q_{n,i+1} = 1.$$

The following theorem is found in [5].

Theorem 5. *Terms on the same level of the Stern–Brocot tree that are equidistant from either end are reciprocals of each other. Such terms are styled* symmetric complements. *Algebraically,*

$$s_{n,k} = q_{n,2^{n-1}-k+1}.$$

Corollary 6. *Terms that are equidistant from either end of any Stern–Brocot sequence are reciprocals of each other. That is,*

$$\begin{aligned} H_n &= \left\langle \frac{s_{n,1}}{q_{n,1}} = \frac{0}{1}, \frac{s_{n,2}}{q_{n,2}}, \cdots, \frac{s_{n,2^n}}{q_{n,2^n}}, \frac{s_{n,2^n+1}}{q_{n,2^n+1}} = \frac{1}{0} \right\rangle \\ &= \left\langle \frac{s_{n,1}}{q_{n,1}}, \frac{s_{n,2}}{q_{n,2}}, \cdots, \frac{s_{n,2^{n-1}}}{q_{n,2^{n-1}}}, \frac{s_{n,2^{n-1}+1}}{q_{n,2^{n-1}+1}} = \frac{1}{1}, \frac{q_{n,2^{n-1}}}{s_{n,2^{n-1}}}, \cdots, \frac{q_{n,2}}{s_{n,2}}, \frac{q_{n,1}}{s_{n,1}} \right\rangle. \end{aligned} \tag{1}$$

Proof. The proof proceeds by induction on n. Our result is true for $n = 1$. Suppose it is true for some H_k.

By Definition 1 consecutive odd-subscripted terms in H_{k+1} represent consecutive terms in H_k, and so for these terms our result is true by our inductive hypothesis.

By Definition 3, consecutive even-subscripted terms in H_{k+1} represent consecutive terms from level $k+1$ of the Stern–Brocot tree. For these terms our result is true by Theorem 5.

Thus our result is true for H_{k+1}. The result follows. □

Definition 7. (Cross-differences) The *cross-difference* of r/m and c/d is $mc-rd$.

Theorem 4 reveals that all cross-differences of consecutive terms in H_n have cross-difference 1.

Definition 8. (Stern–Brocot Cross-Differences) For $i = 1, 2, \ldots, 2^{n-1}-1$, and $n > 1$, $\mathbb{C}_{n,i}$, the ith *Stern–Brocot Cross-Difference* in level n of the tree, is given by

$$\mathbb{C}_{n,i} = s_{n,2i+2}q_{n,2i} - s_{n,2i}q_{n,2i+2}$$

where $s_{n,2i}, s_{n,2i+2}, q_{n,2i}$ and $q_{n,2i+2}$ are terms defined in Definition 1.

The following theorem, found in [5], shows that the ith cross-difference in a level of the tree takes the value $2j_i+3$, where j_i is the number of factors that have value 2 in the prime factorization of i. Note that $\mathbb{C}_{n,i}$ is only dependent on i.

Theorem 9. *Let* $i = 2^{j_i}(2m_i - 1)$. *Then*

$$\mathbb{C}_{n,i} = 2j_i + 3.$$

In what follows we adopt the following notation for continued fractions and their approximants.

$$b_0 + \mathbf{K}\left(\frac{a_n}{b_n}\right) = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\ddots}}}$$

$$= b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \ldots$$

The successive approximants of $b_0 + \mathbf{K}\left(\frac{a_n}{b_n}\right)$ are

$$\frac{A_0}{B_0} = b_0,$$
$$\frac{A_1}{B_1} = b_0 + \frac{a_1}{b_1},$$
$$\frac{A_2}{B_2} = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2}} \text{ and so on.}$$

If $b_0 = 0$, we often denote $b_0 + \mathbf{K}\left(\frac{a_n}{b_n}\right)$ as simply $\mathbf{K}\left(\frac{a_n}{b_n}\right)$. We also designate

$$\Delta_n := A_{n-1}B_n - A_nB_{n-1}. \tag{2}$$

Daniel Bernoulli discovered the following theorem on continued fractions in 1775 [6]. We state the version found in Lorentzen and Waadeland [10] with slight modifications.

Theorem 10. *For $N > 1$, the sequences*

$$\langle A_n \rangle_{n=0}^{N} \text{ and } \langle B_n \rangle_{n=0}^{N}$$

of complex numbers are the canonical numerators and denominators respectively of some continued fraction

$$b_0 + \mathbf{K}\left(\frac{a_n}{b_n}\right)$$

if and only if

1. $\Delta_n \neq 0$ *for* $n \geq 1$ *and*
2. $B_0 = 1$.

Then

$$b_0 + \mathbf{K}\left(\frac{a_n}{b_n}\right)$$

is uniquely determined by

$$b_0 := A_0$$

$$b_1 := B_1 \qquad a_1 := A_1 - A_0B_1$$

$$b_n := \tfrac{A_{n-2}B_n - B_{n-2}A_n}{\Delta_{n-1}} \text{ for } n \geq 2 \qquad a_n := -\tfrac{\Delta_n}{\Delta_{n-1}} \text{ for } n \geq 2$$

where $\Delta_n = \prod_{k=1}^{n} (-a_k)$.

Remark 11. Theorem 10 tells us that a pair of sequences

$$U = \langle u_0, u_1, u_2, \ldots, u_N \rangle \text{ and } V = \langle v_0, v_1, v_2, \ldots, v_N \rangle$$

forms approximants, u_n/v_n, of some continued fraction provided that U and V abide by the following necessary and sufficient conditions for the existence of this continued fraction:

1. $u_{n-1}v_n - u_nv_{n-1} \neq 0$, for $n \geq 1$,
2. $v_0 = 1$.

If these conditions are satisfied, then a continued fraction can be formed from U and V possessing the following properties:

$$b_0 = u_0$$

$$b_1 = v_1 \qquad a_1 = u_1 - u_0 v_1$$

$$b_n = \tfrac{u_{n-2}v_n - v_{n-2}u_n}{u_{n-1}v_n - u_n v_{n-1}} \text{ for } n \geq 2 \qquad a_n = -\tfrac{u_{n-1}v_n - u_n v_{n-1}}{u_{n-2}v_{n-1} - u_{n-1}v_{n-2}} \text{ for } n \geq 2.$$

2. The Stern–Brocot Continued Fraction

We are now able to prove our main result which allows us to represent the Stern–Brocot sequence as successive approximants of a continued fraction.

Theorem 12 (Stern–Brocot continued fraction). *For $n > 0$, let*

$$H_n = \left\langle \frac{s_{n,1}}{q_{n,1}}, \frac{s_{n,2}}{q_{n,2}}, \cdots, \frac{s_{n,2^n+1}}{q_{n,2^n+1}} \right\rangle$$

be the Stern–Brocot sequence of order n, where

$$\frac{s_{n,1}}{q_{n,1}} = \frac{0}{1}, \frac{s_{n,2}}{q_{n,2}} = \frac{1}{n} \text{ and } \frac{s_{n,2^n+1}}{q_{n,2^n+1}} = \frac{1}{0}.$$

Let also j_k represent the number of factors having value 2 in the prime factorization of k. Then

a) for $i = 1, 2, \cdots, 2^{n-1}+1$, the term $s_{n,i}/q_{n,i}$ is the $(i-1)$th approximant of the continued fraction,

$$\mathbb{H}_n = \frac{1}{n} \; - \; \frac{1}{2j_1+1} \; - \; \frac{1}{2j_2+1} \; - \; \frac{1}{2j_3+1} \; - \; \cdots \; - \; \frac{1}{2j_{2^{n-1}-1}+1}, \quad (3)$$

b) for $i = 2^{n-1}+2, \cdots, 2^n+1$, the term $s_{n,i}/q_{n,i}$ is the reciprocal of the $(2^n - i + 1)$th approximant of the continued fraction at (3).

Proof. We prove each part separately.

a) Our first objective is to show that

$$\langle s_{n,1}, s_{n,2}, \cdots, s_{n,2^n+1} \rangle \text{ and } \langle q_{n,1}, q_{n,2}, \cdots, q_{n,2^n+1} \rangle$$

are two sequences for which $s_{n,i}/q_{n,i}$, where $i = 1, 2, \cdots, 2^{n-1}+1$, represents the $(i-1)$th approximant of some continued fraction. Our second objective is to show that this continued fraction must be (3).

The two necessary and sufficient conditions in Theorem 10 (and Remark 11) for establishing the existence of our continued fraction are satisfied in

$$\langle s_{n,1}, s_{n,2}, \cdots, s_{n,2^n+1} \rangle \text{ and } \langle q_{n,1}, q_{n,2}, \cdots, q_{n,2^n+1} \rangle,$$

since by

i) Theorem 4,

$$s_{n,i-1}q_{n,i} - s_{n,i}q_{n,i-1} = -1 \neq 0, \text{ and by}$$

ii) Definition 1,

$$\frac{s_{n,1}}{q_{n,1}} = \frac{0}{1},$$

that is, $q_{n,1} = 1$.

Thus our first objective has been achieved. Accordingly, by Theorems 4 and 10, and using our continued fraction notation, the continued fraction that we seek has the following properties:

$b_0 = s_{n,1} = 0$

$b_1 = q_{n,2} = n$ $\qquad$ $a_1 = s_{n,2} - s_{n,1}q_{n,2} = 1$

$b_i = \frac{s_{n,i-2}q_{n,i} - q_{n,i-2}s_{n,i}}{-1}$ for $i \geq 2$ $\qquad$ $a_i = -\frac{s_{n,i-1}q_{n,i} - s_{n,i}q_{n,i-1}}{s_{n,i-2}q_{n,i-1} - s_{n,i-1}q_{n,i-2}} = -1$, for $i \geq 2$.

Consider b_i, where $i \geq 2$. Let $m \in \mathbb{N}$. There are two cases.

i) i odd $(= 2m + 1)$: Then b_i represents the mth cross-difference in any level of the tree. By Theorem 9, we have

$$b_i = 2j_m + 3$$

and since

$$2j_m + 3 = 2j_{2m} + 1,$$

it follows that

$$b_i = b_{2m+1} = 2j_{2m} + 1.$$

ii) i even $(= 2m)$: By Definition 3, odd-subscripted terms in the Stern–Brocot sequence of order n are consecutive terms in the Stern–Brocot sequence of order $n - 1$. Accordingly, by Theorem 4,

$$b_i = b_{2m} = 1.$$

Since

$$2j_{2m-1} + 1 = 1,$$

it follows that

$$b_i = b_{2m} = 2j_{2m-1} + 1.$$

Combining both cases, for $i > 1$ the ith term in $\mathbb{H}_n$ is

$$\frac{a_i}{b_i} = \frac{-1}{2j_{i-1} + 1}.$$

The result follows and our second objective has been achieved.

b) The result follows from a) and Corollary 6. □

Note that in Theorem 12,

$$\mathbb{H}_1 = \frac{1}{n}$$

since the term

$$\frac{1}{2j_{2^{n-1}-1}+1}$$

does not exist for $n = 1$.

Corollary 13. *We have*

$$\mathbb{H}_n = \frac{1}{n} \; - \; \frac{1}{1} \; - \; \frac{1}{2j_1+3} \; - \; \frac{1}{1} \; - \; \frac{1}{2j_2+3} \; - \; \cdots \; - \; \frac{1}{2j_{2^{n-2}-1}+3} \; - \; \frac{1}{1}.$$

Proof. Since for $m \in \mathbb{N}$,

$$\frac{1}{2j_{2m}+1} = \frac{1}{2j_m+3} \text{ and}$$
$$\frac{1}{2j_{2m-1}+1} = 1,$$

the result follows by Theorem 12. □

Corollary 14. *The subsequence of terms in* (1) *with even subscripts, that is, the subsequence*

$$\left\langle \frac{s_{n,2}}{q_{n,2}}, \frac{s_{n,4}}{q_{n,4}}, \frac{s_{n,6}}{q_{n,6}}, \cdots, \frac{q_{n,6}}{s_{n,6}}, \frac{q_{n,4}}{s_{n,4}}, \frac{q_{n,2}}{s_{n,2}} \right\rangle,$$

represents level n of the Stern–Brocot tree and is represented by consecutive odd-subscripted approximants of (3)

Proof. By Definitions 1 and 3, and Theorem 12. □

The following corollary shows us an interesting continued fraction expression for unity.

Corollary 15. *For $n > 0$,*

$$1 = \frac{1}{n} \; - \; \frac{1}{2j_1+1} \; - \; \frac{1}{2j_2+1} \; - \; \frac{1}{2j_3+1} \; - \; \cdots \; - \; \frac{1}{2j_{2^{n-1}-1}+1}.$$

Proof. This is the middle term in (1) which corresponds to the last approximant of (3) . Note that for $n = 1$, the term

$$\frac{1}{2j_{2^{n-1}-1}+1}$$

does not exist. □

Example 16. By Theorem 12, for $n = 4$,

$$\mathbb{H}_4 = \frac{1}{4} \; - \; \frac{1}{1} \; - \; \frac{1}{3} \; - \; \frac{1}{1} \; - \; \frac{1}{5} \; - \; \frac{1}{1} \; - \; \frac{1}{3} \; - \; \frac{1}{1}$$

from which we obtain the Stern–Brocot sequence

$$\begin{aligned} H_4 &= \left\langle \frac{s_{n,1}}{q_{n,1}}, \frac{s_{n,2}}{q_{n,2}}, \cdots, \frac{s_{n,8}}{q_{n,8}}, \frac{s_{n,9}}{q_{n,9}} = \frac{1}{1}, \frac{q_{n,8}}{s_{n,8}}, \cdots, \frac{q_{n,2}}{s_{n,2}}, \frac{q_{n,1}}{s_{n,1}} \right\rangle \\ &= \left\langle \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, \frac{2}{1}, \frac{5}{2}, \frac{3}{1}, \frac{4}{1}, \frac{1}{0} \right\rangle. \end{aligned}$$

By Corollary 14, level 4 of the Stern–Brocot tree is

$$\left\langle \frac{1}{4}, \frac{2}{5}, \frac{3}{5}, \frac{3}{4}, \frac{4}{3}, \frac{5}{3}, \frac{5}{2}, \frac{4}{1} \right\rangle.$$

3. Interleaving

The Paperfolding sequence has a dual representation – one based on *interleaving* and the other based on *mirroring* [4]. So too with the sequence of terms in the Stern–Brocot continued fraction. We now explore this interleaving representation.

Definition 17. (Interleave Operator). The *interleave operator* # acting on the two sequences

$$U = \langle u_1, u_2, \ldots, u_k \rangle \text{ and } V = \langle v_1, v_2, \ldots, v_n \rangle$$

where $k > n$, generates the following interleaved sequence:

$$U\#V = \langle u_1, \ldots, u_p, v_1, u_{p+1}, \ldots, u_{2p}, v_2, u_{2p+1}, \ldots, u_{np}, v_n, u_{np+1}, \ldots, u_k \rangle,$$

where $p = \left\lfloor \dfrac{k}{n+1} \right\rfloor$. Also $U\#V$ when de-leaved of U becomes V.

Example 18. Let

$$U = \langle u_1, u_2, \ldots, u_{n+1} \rangle \text{ and } V = \langle v_1, v_2, \ldots, v_n \rangle.$$

Then

$$U\#V = \langle u_1, v_1, u_2, v_2, \ldots, u_n, v_n, u_{n+1} \rangle.$$

Definition 19. (Pervasive Odd-Valued Sequence). The *Pervasive Odd-Valued Sequence*, $\mathbb{P}_{m,k}$, is defined as

$$\mathbb{P}_{m,k} = \left\langle \underbrace{2k-3, 2k-3, \ldots, 2k-3}_{2^{m-1} \text{ terms}} \right\rangle$$

where $m, k \in \mathbb{N}$.

Definition 20. ($\mathbb{T}_n$ sequence). Let $\mathbb{T}_n$ be the sequence of partial denominators $\langle b_i \rangle$ in $\mathbb{H}_n$ for $i = 2, 3, \ldots, 2^{n-1}$ and $n > 1$.

Theorem 21 (Interleaving representation of the Stern–Brocot continued fraction). *For $i > 0, n > 1$, the sequence of partial denominators $\langle b_i \rangle$ in $\mathbb{H}_n$ is*

$$\langle n, \mathbb{P}_{n-1,2} \# \ldots \# \mathbb{P}_{2,n-1} \# \mathbb{P}_{1,n} \rangle .$$

Proof. From Theorem 12, $b_1 = n$. From (3) , every odd-placed term in $\mathbb{T}_n$ is of the form $2j_{_{odd}} + 1$. Since $j_{odd} = 0$, we have

$$2j_{_{odd}} + 1 = 1.$$

There are 2^{n-2} odd-placed terms in $\mathbb{T}_n$, beginning with the first term and ending with the last term in $\mathbb{T}_n$. Hence $\mathbb{T}_n$ has been formed through an interleave $\mathbb{P}_{n-1,2}$ applied to every other term in $\mathbb{T}_n$.

Let $\mathbb{T}_n^{(1)}$ be the residue of $\mathbb{T}_n$ once it has been de-leaved of $\mathbb{P}_{n-1,2}$. Every odd-placed term in $\mathbb{T}_n^{(1)}$ is of the form

$$2j_{_{2(2m-1)}} + 1.$$

Since for every m,

$$2j_{_{2(2m-1)}} + 1 = 3$$

and there are 2^{n-3} of these terms in $\mathbb{T}_n^{(1)}$ we can de-leave $\mathbb{T}_n^{(1)}$ by $\mathbb{P}_{n-2,3}$ to form its residue $\mathbb{T}_n^{(2)}$.

But every odd-placed term in $\mathbb{T}_n^{(2)}$ is

$$2j_{_{2^2(2m-1)}} + 1 = 5$$

and there are 2^{n-4} of these terms in $\mathbb{T}_n^{(2)}$. Hence, we can de-leave $\mathbb{T}_n^{(2)}$ by $\mathbb{P}_{n-3,4}$ to form its residue $\mathbb{T}_n^{(3)}$.

Proceeding in this way we are eventually left with one term $\mathbb{P}_{1,n}$. The result follows. □

Example 22. Find $\mathbb{H}_4$ by interleaving Answer: For $n = 4$,

$$\begin{aligned} \mathbb{P}_{1,4} &= 5 \\ \mathbb{P}_{2,3} \# \mathbb{P}_{1,4} &= 3, \quad 5, \quad 3 \\ \mathbb{P}_{3,2} \# \mathbb{P}_{2,3} \# \mathbb{P}_{1,4} &= 1, \quad 3, \quad 1, \quad 5, \quad 1, \quad 3, \quad 1 \end{aligned}$$

and so the sequence of partial denominators $\langle b_i \rangle$ in $\mathbb{H}_4$ is

$$\langle 4, 1, 3, 1, 5, 1, 3, 1 \rangle .$$

Therefore

$$\mathbb{H}_4 = \frac{1}{4} - \frac{1}{1} - \frac{1}{3} - \frac{1}{1} - \frac{1}{5} - \frac{1}{1} - \frac{1}{3} - \frac{1}{1}.$$

Corollary 23. $\mathbb{T}_n$ *is palindromic.*

Proof. From Theorem 21,

$$\mathbb{T}_n = \langle \mathbb{P}_{n-1,2} \# \ldots \# \mathbb{P}_{2,n-1} \# \mathbb{P}_{1,n} \rangle .$$

The result follows from Definitions 17 and 19. □

4. Mirroring

We now offer a method for determining $\mathbb{H}_{n+1}$ from $\mathbb{H}_n$. It is called mirroring because we mirror part of the continued fraction around a central term.

Theorem 24 (Mirroring of $\mathbb{T}_n$). *We have*

$$\mathbb{T}_n = \langle \mathbb{T}_{n-1}, 2n-3, \mathbb{T}_{n-1} \rangle .$$

Proof. For $n > 2$,

$$\begin{aligned} \mathbb{T}_n &= \langle \mathbb{P}_{n-1,2} \# \ldots \# \mathbb{P}_{2,n-1} \# \mathbb{P}_{1,n} \rangle \\ &= \langle \mathbb{P}_{n-2,2} \# \ldots \# \mathbb{P}_{1,n-1}, \mathbb{P}_{1,n}, \mathbb{P}_{n-2,2} \# \ldots \# \mathbb{P}_{1,n-1} \rangle \\ &= \langle \mathbb{T}_{n-1}, 2n-3, \mathbb{T}_{n-1} \rangle . \end{aligned}$$

For $n = 2$, we have $\mathbb{T}_2 = \langle 1 \rangle$ which we designate as $\langle \mathbb{T}_1, 1, \mathbb{T}_1 \rangle$. □

Theorem 25 (Mirroring of the Stern–Brocot continued fraction). *For $n > 1$, let*

$$\begin{aligned} \mathbb{H}_n &= \frac{1}{n - \omega_n}, \text{ where} \\ \omega_n &= \frac{1}{2j_1+1} - \frac{1}{2j_2+1} - \frac{1}{2j_3+1} - \cdots - \frac{1}{2j_{2^{n-1}-1}+1}. \end{aligned}$$

Then

$$\begin{aligned} \mathbb{H}_{n+1} &= \frac{1}{(n+1) - \omega_{n+1}}, \text{ where} \\ \omega_{n+1} &= \omega_n - \frac{1}{2n-1} - \omega_n. \end{aligned}$$

Proof. For $i > 0, n > 1$, the sequence of partial denominators $\langle b_i \rangle$ in $\mathbb{H}_n$ is $\langle n, \mathbb{T}_n \rangle$. From Theorems 21 and 24, for $i > 0, n > 1$, the sequence of partial denominators $\langle b_i \rangle$ in $\mathbb{H}_{n+1}$ is

$$\langle n+1, \mathbb{T}_{n+1} \rangle = \langle n+1, \mathbb{T}_n, 2(n+1)-3, \mathbb{T}_n \rangle = \langle n+1, \mathbb{T}_n, 2n-1, \mathbb{T}_n \rangle .$$

The result follows. □

Remark 26. Theorem 25 tells us that if we know $\mathbb{H}_n$ then all we need to do to discover $\mathbb{H}_{n+1}$ is follow three simple steps:

- first add 1 to the denominator of the first term in $\mathbb{H}_n$,
- then suffix a new term
$$\frac{-1}{2n-1},$$
- then suffix the terms represented by $-\omega_n$.

Example 27. Find $\mathbb{H}_4$ by mirroring of $\mathbb{H}_3$. Answer: From Theorem 25,
$$\mathbb{H}_3 = \frac{1}{3-\omega_3} = \frac{1}{3} \;-\; \frac{1}{1} \;-\; \frac{1}{3} \;-\; \frac{1}{1}.$$
That is,
$$\omega_3 = \frac{1}{1} \;-\; \frac{1}{3} \;-\; \frac{1}{1}.$$
Hence,
$$\omega_4 = \omega_3 \;-\; \frac{1}{3} \;-\; \omega_3.$$
And so,
$$\begin{aligned}\mathbb{H}_4 &= \frac{1}{4-\omega_4} \\ &= \frac{1}{4} \;-\; \omega_3 \;-\; \frac{1}{5} \;-\; \omega_3 \\ &= \frac{1}{4} \;-\; \frac{1}{1} \;-\; \frac{1}{3} \;-\; \frac{1}{1} \;-\; \frac{1}{5} \;-\; \frac{1}{1} \;-\; \frac{1}{3} \;-\; \frac{1}{1}.\end{aligned}$$
Equivalently, repeatedly using Theorem 24,
$$\mathbb{T}_4 = \langle \mathbb{T}_3, 5, \mathbb{T}_3 \rangle = \langle \mathbb{T}_2, 3, \mathbb{T}_2, 5, \mathbb{T}_2, 3, \mathbb{T}_2 \rangle = \langle 1, 3, 1, 5, 1, 3, 1 \rangle$$
Accordingly the sequence of partial denominators $\langle b_i \rangle$ in $\mathbb{H}_4$ is
$$\langle 4, 1, 3, 1, 5, 1, 3, 1 \rangle$$
and so
$$\mathbb{H}_4 = \frac{1}{4} \;-\; \frac{1}{1} \;-\; \frac{1}{3} \;-\; \frac{1}{1} \;-\; \frac{1}{5} \;-\; \frac{1}{1} \;-\; \frac{1}{3} \;-\; \frac{1}{1}.$$

Example 28. What is the 3rd entry in the 5th level of the Stern–Brocot tree? Answer: The 3rd entry in the 5th level of the tree is the $(2 \cdot 3 - 1)$th, that is, the 5th approximant of $\mathbb{H}_5$. By Theorem 24, using mirroring,
$$\begin{aligned}\mathbb{T}_2 &= \langle 1 \rangle \\ \mathbb{T}_3 &= \langle 1, 3, 1 \rangle \\ \mathbb{T}_4 &= \langle 1, 3, 1, 5, 1, 3, 1 \rangle .\end{aligned}$$

From Theorem 21, the 5th approximant of $\mathbb{H}_5$ is

$$\frac{1}{5} - \frac{1}{1} - \frac{1}{3} - \frac{1}{1} - \frac{1}{5} = \frac{3}{8}.$$

Notice that we did not have to produce $\mathbb{T}_5$ in this example. We only needed to find the first n for which $\mathbb{T}_n$ has cardinality at least 4. This we obtained with $\mathbb{T}_4$.

5. Diagonals

Continued fraction expansions exist to represent branches and diagonals [3], but these expansions are not based on the Stern–Brocot continued fraction. We now apply the Stern–Brocot continued fraction to determine continued fraction expressions for diagonals in the tree.

Definition 29. (Left and Right Diagonals). We make the following definitions:
Left diagonals: The sequence of kth terms **from the left**, of successive levels of the Stern–Brocot tree is called the kth left diagonal and is represented as $\mathbb{L}_k$.

Right diagonals: The sequence of kth terms **from the right**, of successive levels of the Stern–Brocot tree is called the kth right diagonal and is represented as $\mathbb{R}_k$.

Theorem 30 (Diagonal sequences). *For $t = 1, 2, 3, \ldots$*

$$\mathbb{L}_k = \left\langle \frac{1}{\lceil \log_2 k \rceil + t - \omega_k} \right\rangle_{t=1}^{\infty}$$
$$\mathbb{R}_k = \langle \lceil \log_2 k \rceil + t - \omega_k \rangle_{t=1}^{\infty}$$

where

$$\omega_k = \frac{1}{2j_1 + 1} - \frac{1}{2j_2 + 1} - \frac{1}{2j_3 + 1} - \cdots - \frac{1}{2j_{2(k-1)} + 1}.$$

Proof. We have

i) $\mathbb{L}_k$: By Definition 1, the first appearance of a kth term, from the left in a level of the tree, appears in level

$$\lceil \log_2 k \rceil + 1.$$

Let p/q be the second term in $\mathbb{L}_k$. Then by Definition 1, p/q is the first appearance of a kth term, from the left of a level, in the left half of the tree. It therefore appears in level

$$\lceil \log_2 k \rceil + 2.$$

By Definition 3, the kth term in this level must therefore be the $(2k-2)$th approximant of

$$\mathbb{H}_{\lceil \log_2 k \rceil + 2}.$$

Then by Theorem 12,

$$\frac{p}{q} = \frac{1}{\lceil \log_2 k \rceil + 2} \; - \; \frac{1}{2j_1+1} \; - \; \frac{1}{2j_2+1} \; - \; \frac{1}{2j_3+1} \; - \; \cdots \; - \; \frac{1}{2j_{2(k-1)}+1},$$

and subsequent terms in $\mathbb{L}_k$ must be, for $t = 1, 2, 3, \cdots$

$$\frac{1}{\lceil \log_2 k \rceil + 2 + t} \; - \; \frac{1}{2j_1+1} \; - \; \frac{1}{2j_2+1} \; - \; \frac{1}{2j_3+1} \; - \; \cdots \; - \; \frac{1}{2j_{2(k-1)}+1}.$$

But Remark 26 tells us that the first term in $\mathbb{L}_k$ must then be

$$\frac{1}{\lceil \log_2 k \rceil + 1} \; - \; \frac{1}{2j_1+1} \; - \; \frac{1}{2j_2+1} \; - \; \frac{1}{2j_3+1} \; - \; \cdots \; - \; \frac{1}{2j_{2(k-1)}+1}.$$

The result follows.

ii) $\mathbb{R}_k$: By Theorem 5, the element of $\mathbb{R}_k$ in level m is the reciprocal of the element of $\mathbb{L}_k$ in level m. The result follows. □

Example 31. For $t = 1, 2, 3, \cdots$ and $k = 3$, we have

$$\omega_3 = \frac{1}{1} \; - \; \frac{1}{3} \; - \; \frac{1}{1} \; - \; \frac{1}{5} = \frac{7}{3}.$$

Thus

$$\begin{aligned} \mathbb{L}_3 &= \left\langle \frac{1}{\lceil \log_2 3 \rceil + t - \frac{7}{3}} \right\rangle_{t=1}^{\infty} \\ &= \left\langle \frac{3}{3t-1} \right\rangle_{t=1}^{\infty} \\ &= \left\langle \frac{3}{2}, \frac{3}{5}, \frac{3}{8}, \frac{3}{11}, \cdots \right\rangle \end{aligned}$$

and

$$\begin{aligned} \mathbb{R}_3 &= \left\langle \frac{3t-1}{3} \right\rangle_{t=1}^{\infty} \\ &= \left\langle \frac{2}{3}, \frac{5}{3}, \frac{8}{3}, \frac{11}{3}, \cdots \right\rangle. \end{aligned}$$

Corollary 32. *For $t \in \mathbb{N}$, $n > 1$, the first $2n-1$ terms in*

$$\left\langle \lceil \log_2 n \rceil + t, \mathbb{T}_{\lceil \log_2 n \rceil + 2} \right\rangle$$

represents the sequence of partial denominators $\langle b_i \rangle$ in the continued fraction of the tth entry of $\mathbb{L}_n$.

Proof. The result follows from Theorem 30 and Definition 20 where we note that $\mathbb{T}_k$ has cardinality $2^{k-1} - 1$ and $\lceil \log_2 n \rceil + 2$ is the smallest value of k for which $2^{k-1} - 1 \geq 2n - 1$. □

6. Branches

The Stern–Brocot tree possesses branches. In fact, the entire tree could be depicted as a series of branches whereby each term, except 0/1 and 1/0, appears on both a left and a right branch. Branches in the left half of the tree are analogous to clusters in the iterated Gauss map [1]. This suggests that features of the Gauss map can be derived through an understanding of Stern–Brocot branches. Our goal is to determine a continued fraction whose successive approximants represent successive terms in branches of the Stern–Brocot tree. First we require some definitions and results. Proofs for results can be found at [5].

Definition 33. (Left and Right Branches). In the Stern–Brocot tree, the set of all terms possessing a common parent μ in which all terms are

1. smaller than μ, is called the *left branch* of μ, and is represented as $\mathcal{B}_{L(\mu)}$;
2. greater than μ, is called the *right branch* of μ, and is represented as $\mathcal{B}_{R(\mu)}$

We also define the *augmented* left and right branches respectively, as

$$\mathcal{B}'_{L(\mu)} = \left\langle \frac{0}{1}, \mathcal{B}_{L(\mu)} \right\rangle \text{ and } \mathcal{B}'_{R(\mu)} = \left\langle \frac{0}{1}, \mathcal{B}_{R(\mu)} \right\rangle.$$

It follows that each term in the tree, except for those found in level 0, belongs to two branches - the left branch of one parent and the right branch of the other parent. The following theorem is found in [5].

Theorem 34 (Branch sequences). *Let r/m and c/d, where $r/m < c/d$, be the parents of μ. Then for $i = 1, 2, 3, \ldots$,*

i) the left branch of μ is

$$\mathcal{B}_{L(\mu)} = \left\langle \frac{(i+1)\,r + ic}{(i+1)\,m + id} \right\rangle_{i=1}^{\infty},$$

ii) the right branch of μ is

$$\mathcal{B}_{R(\mu)} = \left\langle \frac{ir + (i+1)\,c}{im + (i+1)\,d} \right\rangle_{i=1}^{\infty}.$$

Example 35. The parents of 5/8 are 3/5 and 2/3. Accordingly,

$$\mathcal{B}_{L\left(\frac{5}{8}\right)} = \left\langle \frac{(i+1)\,3 + i2}{(i+1)\,5 + i3} \right\rangle_{i=1}^{\infty} = \left\langle \frac{8}{13}, \frac{13}{21}, \frac{18}{29}, \frac{23}{37}, \cdots \right\rangle$$

and

$$\mathcal{B}_{R\left(\frac{5}{8}\right)} = \left\langle \frac{i3 + (i+1)\,2}{i5 + (i+1)\,3} \right\rangle_{i=1}^{\infty} = \left\langle \frac{7}{11}, \frac{12}{19}, \frac{17}{27}, \frac{22}{35}, \cdots \right\rangle.$$

Corollary 36. *Terms in $\mathcal{B}_{L(\mu)}$ and $\mathcal{B}_{R(\mu)}$ have limit μ.*

Proof. By Theorem 34,

$$\lim_{i\to\infty}\frac{(i+1)\,r+ic}{(i+1)\,m+id}=\lim_{i\to\infty}\frac{ir+(i+1)\,c}{im+(i+1)\,d}=\frac{r+c}{m+d}=\mu.$$

□

We now discover a continued fraction whose approximants represent successive terms in left branches and another similar continued fraction whose approximants represent successive terms in right branches.

Theorem 37 (Branch continued fraction). *Let r/m and c/d, where $r/m<c/d$, be the parents of μ. Then for $i>0$, we have the following:*

i) The ith term in the left branch of μ represents the ith approximant of the continued fraction

$$\frac{2r+c}{2m+d}\;-\;\frac{\frac{1}{2r+c}}{\frac{3r+2c}{2r+c}}\;-\;\frac{1}{2}\;-\;\frac{1}{2}\;-\;\frac{1}{2}\;-\;\cdots \tag{4}$$

ii) The ith term in the right branch of μ represents the ith approximant of the continued fraction

$$\frac{r+2c}{m+2d}\;+\;\frac{\frac{1}{r+2c}}{\frac{2r+3c}{r+2c}}\;-\;\frac{1}{2}\;-\;\frac{1}{2}\;-\;\frac{1}{2}\;-\;\cdots \tag{5}$$

Proof. We prove the result for left branches. The proof for right branches proceeds identically. Let

$$\mathcal{B}'_{L(\mu)}=\left\langle\frac{A_i}{B_i}\right\rangle_{i=0}^{\infty}$$

where, from Definition 2 and Theorem 34,

$$\langle A_i\rangle_{i=0}^{\infty}=\langle 0,2r+c,3r+2c,4r+3c,\ldots\rangle \text{ and}$$
$$\langle B_i\rangle_{i=0}^{\infty}=\langle 1,2m+d,3m+2d,4m+3d,\ldots\rangle\,.$$

We now show that

$$\langle A_i\rangle_{i=0}^{\infty}\text{ and }\langle B_i\rangle_{i=0}^{\infty}$$

satisfy the two necessary and sufficient conditions of Theorem 10 that allow us to formulate a continued fraction whose approximants represent $\mathcal{B}'_{L(\mu)}$. If μ is on level n, say, of the tree, then by Definition 2, r/m and c/d must be consecutive terms in H_{n-1}. Thus by Theorem 4, $rd-mc=-1$, and by (2),

$$\text{i) }\Delta_i=\begin{cases}-(2r+c)\,, & \text{for } i=1\\ -1, & \text{for } i>1\end{cases}\text{, and}$$
$$\text{ii) } B_0=1.$$

Accordingly, using our continued fraction notation, we have by Theorem 10,

$$b_0 = A_0 = 0,$$

$$b_1 = B_1 = 2m + d, \qquad a_1 = A_1 - A_0 B_1 = 2r + c,$$

$$b_i = \tfrac{A_{i-2}B_i - B_{i-2}A_i}{\Delta_{i-1}} \text{ for } i \geq 2, \qquad a_i = -\tfrac{\Delta_i}{\Delta_{i-1}} \text{ for } i \geq 2$$

That is,
$$b_i = \begin{cases} \frac{3r+2c}{2r+c}, & \text{for } i = 2 \\ 2, & \text{for } i > 2 \end{cases}$$

That is,
$$a_i = \begin{cases} \frac{-1}{2r+c}, & \text{for } i = 2 \\ -1, & \text{for } i > 2 \end{cases}.$$

Thus successive approximants of the continued fraction

$$\frac{2r+c}{2m+d} \; - \; \frac{\frac{1}{2r+c}}{\frac{3r+2c}{2r+c}} \; - \; \frac{1}{2} \; - \; \frac{1}{2} \; - \; \frac{1}{2} \; - \; \cdots$$

represent successive terms in $\mathcal{B}'_{L(\mu)}$. The result follows. □

Example 38. The terms 3/5 and 2/3 are the parents of 5/8. By Theorem 37, for $i > 0$, the ith term in the left branch of 5/8 is the ith approximant of the continued fraction

$$\frac{8}{13} \; - \; \frac{\frac{1}{8}}{\frac{13}{8}} \; - \; \frac{1}{2} \; - \; \frac{1}{2} \; - \; \frac{1}{2} \; - \; \cdots$$

That is,

$$\frac{A_1}{B_1} = \frac{8}{13}$$

$$\frac{A_2}{B_2} = \frac{8}{13 - \frac{1}{13}} = \frac{13}{21}$$

$$\frac{A_3}{B_3} = \frac{8}{13 - \frac{\frac{1}{8}}{\frac{13}{8} - \frac{1}{2}}} = \frac{18}{29}$$

$$\frac{A_4}{B_4} = \frac{8}{13 - \frac{\frac{1}{8}}{\frac{13}{8} - \frac{1}{2 - \frac{1}{2}}}} = \frac{23}{37}$$

and so on. These correspond to terms in $\mathcal{B}_{L\left(\frac{5}{8}\right)}$ (see Example 35).

Similarly, for $i > 0$, the ith term in the right branch of 5/8 is the ith approximant of the continued fraction

$$\frac{7}{11} \; + \; \frac{\frac{1}{7}}{\frac{12}{7}} \; - \; \frac{1}{2} \; - \; \frac{1}{2} \; - \; \frac{1}{2} \; - \; \cdots$$

That is,

$$\begin{aligned}
\frac{A_1}{B_1} &= \frac{7}{11} \\
\frac{A_2}{B_2} = \frac{7}{11+\frac{1}{12}} &= \frac{12}{19} \\
\frac{A_3}{B_3} = \frac{7}{11+\frac{\frac{1}{7}}{\frac{12}{7}-\frac{1}{2}}} &= \frac{17}{27} \\
\frac{A_4}{B_4} = \frac{7}{11+\frac{\frac{1}{7}}{\frac{12}{7}-\frac{1}{2-\frac{1}{2}}}} &= \frac{22}{35}
\end{aligned}$$

and so on. These correspond to terms in $\mathcal{B}_{R\left(\frac{5}{8}\right)}$ (see Example 35).

Corollary 39. *Let r/m and c/d be consecutive terms in a Stern–Brocot sequence. Then*

$$\frac{r+c}{m+d}$$

can be represented through two similar continued fractions:

$$\begin{aligned}
&i)\ \frac{r+c}{m+d} = \frac{2r+c}{2m+d} - \frac{\frac{1}{2r+c}}{\frac{3r+2c}{2r+c}} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \cdots, \\
&ii)\ \frac{r+c}{m+d} = \frac{r+2c}{m+2d} + \frac{\frac{1}{r+2c}}{\frac{2r+3c}{r+2c}} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \cdots
\end{aligned}$$

Proof. Let μ have parents r/m and c/d, such that $\mu = \frac{r+c}{m+d}$. The result follows by Theorem 37 and Corollary 36. □

Corollary 40. *Let r/m and c/d, where $r/m < c/d$, be the parents of μ. Then we have the following:*

i) Successive approximants of the following continued fraction expansions are identical

$$\begin{aligned}
&a)\ \frac{1}{2r+c}\left(2m+d - \frac{\frac{1}{2r+c}}{\frac{3r+2c}{2r+c}} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \cdots\right) \\
&b)\ \frac{2m+d}{2r+c} + \frac{\frac{1}{2m+d}}{\frac{3m+2d}{2m+d}} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \cdots
\end{aligned}$$

ii) Successive approximants of the following continued fraction expansions are identical

$$a)\ \frac{1}{r+2c}\left(m+2d+\frac{\frac{1}{r+2c}}{\frac{2r+3c}{r+2c}}\ -\ \frac{1}{2}\ -\ \frac{1}{2}\ -\ \frac{1}{2}\ -\ \cdots\right)$$

$$b)\ \frac{m+2d}{r+2c}\ -\ \frac{\frac{1}{m+2d}}{\frac{2m+3d}{m+2d}}\ -\ \frac{1}{2}\ -\ \frac{1}{2}\ -\ \frac{1}{2}\ -\ \cdots$$

iii) All continued fraction expansions in i) and ii) are equivalent to $1/\mu$.

Proof. We have

i) a): By Theorem 5 the right branch of $1/\mu$ is formed from reciprocals of successive terms in the left branch of μ. Thus the reciprocal of (4) is the continued fraction whose successive approximants represent the right branch of $1/\mu$.

i) b) is the right branch of $1/\mu$ by (5) where we have substituted d/c and m/r as consecutive terms in a Stern–Brocot sequence such that

$$\frac{1}{\mu}=\frac{d+m}{c+r}.$$

ii) a): By Theorem 5 the left branch of $1/\mu$ is formed from reciprocals of successive terms in the right branch of μ. Thus the reciprocal of (5) is the continued fraction whose successive approximants represent the left branch of $1/\mu$.

ii) b) is the left branch of $1/\mu$ by (4) where we have substituted d/c and m/r as consecutive terms in a Stern–Brocot sequence such that

$$\frac{1}{\mu}=\frac{d+m}{c+r}.$$

iii) All the expressions in i) and ii) refer to either right or left branches of $1/\mu$. Each of their limits is $1/\mu$ by Corollary 36. □

7. Offset Branches

We now explore generalized branches of the Stern–Brocot tree - one for which left and right branches are particular cases. These are called *offset* branches and our branches developed in the previous section are found to be offset branches possessing zero offset. Where proofs of results are not shown, these are given at [5].

Definition 41. (Offset Branches). Let $\mathcal{B}_{L(\mu)}$ and $\mathcal{B}_{R(\mu)}$ denote left and right branches respectively of some term μ in the Stern–Brocot tree.

1. For each term in $\mathcal{B}_{L(\mu)}$, locate a term found t **left** movements away. The set of all these new terms is designated $\mathcal{B}_{L(\mu),t}$. We call $\mathcal{B}_{L(\mu),t}$ the *left branch with offset* t *of* μ.

2. For each term in $\mathcal{B}_{R(\mu)}$, locate a term found t **right** movements away. The set of all these new terms is designated $\mathcal{B}_{R(\mu),t}$. We call $\mathcal{B}_{R(\mu),t}$ the *right branch with offset t of μ.*

We also define the *augmented* left and right branches with offset t of μ, respectively, as

$$\mathcal{B}'_{L(\mu),t} = \left\langle \frac{0}{1}, \mathcal{B}_{L(\mu),t} \right\rangle \text{ and } \mathcal{B}'_{R(\mu),t} = \left\langle \frac{0}{1}, \mathcal{B}_{R(\mu),t} \right\rangle.$$

It follows from Definition 41 that

$$\mathcal{B}_{L(\mu)} = \mathcal{B}_{L(\mu),0}, \text{ and } \mathcal{B}_{R(\mu)} = \mathcal{B}_{R(\mu),0}.$$

Example 42.

$$\mathcal{B}_{L\left(\frac{1}{2}\right),2} = \frac{1}{5}, \frac{4}{11}, \frac{7}{17}, \ldots \text{ and } \mathcal{B}_{R\left(\frac{1}{2}\right),2} = \frac{4}{5}, \frac{7}{11}, \frac{10}{17}, \ldots$$

The following theorem is found in [5].

Theorem 43. *Let r/m and c/d be the parents of μ, with $r/m < c/d$. Then*

$$\mathcal{B}_{L(\mu),t} = \left\langle \frac{(ti+i+1)\,r + (ti+i-t)\,c}{(ti+i+1)\,m + (ti+i-t)\,d} \right\rangle_{i=1}^{\infty} \text{ and}$$
$$\mathcal{B}_{R(\mu),t} = \left\langle \frac{(ti+i-t)\,r + (ti+i+1)\,c}{(ti+i-t)\,m + (ti+i+1)\,d} \right\rangle_{i=1}^{\infty}.$$

Corollary 44. *The terms in $\mathcal{B}_{L(\mu),t}$ and $\mathcal{B}_{R(\mu),t}$ have limit μ.*

Proof. By Theorem 43,

$$\begin{aligned} \lim_{i\to\infty} \frac{(ti+i+1)\,r + (ti+i-t)\,c}{(ti+i+1)\,m + (ti+i-t)\,d} &= \lim_{i\to\infty} \frac{(ti+i-t)\,r + (ti+i+1)\,c}{(ti+i-t)\,m + (ti+i+1)\,d} \\ &= \frac{r+c}{m+d} \\ &= \mu. \end{aligned}$$

□

Definition 45. (Intra-Branch Cross-Differences) Let r/m and c/d be the ith and $(i+1)$th elements, respectively, in $\mathcal{B}_{L(\mu),t}$ $\left(\mathcal{B}_{R(\mu),t}\right)$. The ith *Intra-Branch Cross-Difference* of $\mathcal{B}_{L(\mu),t}$ $\left(\mathcal{B}_{R(\mu),t}\right)$, denoted by $\mathbb{D}_{L(\mu),t,i}$ $\left(\mathbb{D}_{R(\mu),t,i}\right)$, is given by $mc - rd$.

The following theorem is found in [5].

Theorem 46. *For all i and μ,*

$$\mathbb{D}_{L(\mu),t,i} = \mathbb{D}_{R(\mu),t,i} = (t+1)^2 .$$

We are now able to state the main result for offset branches.

Theorem 47 (Offset Branch continued fraction). *Let r/m and c/d be the parents of μ, with $r/m < c/d$. Then for $i > 0$,*

1. *The ith term in $\mathcal{B}_{L(\mu),t}$ is the ith approximant of the continued fraction*

$$\frac{(t+2)\,r+c}{(t+2)\,m+d} \; - \; \frac{\frac{(t+1)^2}{(t+2)r+c}}{\frac{(2t+3)r+(t+2)c}{(t+2)r+c}} \; - \; \frac{1}{2} \; - \; \frac{1}{2} \; - \; \frac{1}{2} \; - \; \cdots ,$$

2. *The ith term in $\mathcal{B}_{R(\mu),t}$ is the ith approximant of the continued fraction*

$$\frac{r+(t+2)\,c}{m+(t+2)\,d} \; + \; \frac{\frac{(t+1)^2}{r+(t+2)c}}{\frac{(t+2)r+(2t+3)c}{r+(t+2)c}} \; - \; \frac{1}{2} \; - \; \frac{1}{2} \; - \; \frac{1}{2} \; - \; \cdots$$

Proof. The proof is similar to that given for Theorem 37. We prove the result for left branches with offset t. The proof for right branches with offset t proceeds identically.

Let

$$\mathcal{B}'_{L(\mu),t} = \left\langle \frac{A_i}{B_i} \right\rangle_{i=0}^{\infty}$$

where, from Definition 41 and Theorem 43,

$$\langle A_i \rangle_{i=0}^{\infty} = \langle 0, (t+2)\,r+c, (2t+3)\,r+(t+2)\,c, \ldots \rangle \text{ and}$$
$$\langle B_i \rangle_{i=0}^{\infty} = \langle 1, (t+2)\,m+d, (2t+3)\,m+(t+2)\,d, \ldots \rangle .$$

We now show that $\langle A_i \rangle_{i=0}^{\infty}$ and $\langle B_i \rangle_{i=0}^{\infty}$ satisfy the two necessary and sufficient conditions of Theorem 10 that allow us to formulate a continued fraction whose approximants represent $\mathcal{B}'_{L(\mu),t}$. If μ is on level n, say, of the tree, then by Definition 2, r/m and c/d must be consecutive terms in H_{n-1}. Thus by Theorem 4,

$$rd - mc = -1.$$

By (2) and Theorem 46,

$$\text{i) } \Delta_i = \begin{cases} -((2+t)\,r+c), & \text{for } i = 1 \\ -(t+1)^2, & \text{for } i > 1 \end{cases}, \text{ and}$$
$$\text{ii) } B_0 = 1.$$

Accordingly, by Theorem 10,

$$b_0 := A_0 = 0,$$

$$b_1 := B_1 = (2+t)\,m + d, \qquad a_1 := A_1 - A_0B_1 = (2+t)\,r + c,$$

$$b_i := \tfrac{A_{i-2}B_i - B_{i-2}A_i}{\Delta_{i-1}} \text{ for } i \geq 2. \qquad a_i := -\tfrac{\Delta_i}{\Delta_{i-1}} \text{ for } i \geq 2.$$

That is,
$$b_i = \begin{cases} \frac{(2t+3)r+(t+2)c}{(t+2)r+c}, & \text{for } i = 2 \\ 2, & \text{for } i > 2 \end{cases}.$$

That is,
$$a_i = \begin{cases} \frac{-(t+1)^2}{(t+2)r+c}, & \text{for } i = 2 \\ -1, & \text{for } i > 2 \end{cases}.$$

Thus successive approximants of the continued fraction

$$\frac{(t+2)\,r+c}{(t+2)\,m+d} \;-\; \frac{\frac{(t+1)^2}{(t+2)r+c}}{\frac{(2t+3)r+(t+2)c}{(t+2)r+c}} \;-\; \frac{1}{2} \;-\; \frac{1}{2} \;-\; \frac{1}{2} \;-\; \cdots$$

represent successive terms in $\mathcal{B}'_{L(\mu),t}$. The result follows. □

It is a simple exercise to extend Corollaries 39 and 40 to obtain results for offset branches.

8. Further Developments

We have shown how the Stern–Brocot tree and sequence can be expressed in terms of a simple continued fraction. We have also extended this idea to include continued fraction expansions for Stern–Brocot branches and diagonals. There are two areas that the interested reader may choose to explore:

- Can these ideas be extended to other binary trees, particularly the Calkin–Wilf tree, or even to $n-$ary trees?
- Since branches in the left half of the Stern–Brocot tree are analogous to clusters in the iterated Gauss map, what parts of the iterated or generalized Gauss map, if any, are analogous to offset branches in the Stern–Brocot tree?

Acknowledgement I am delighted to acknowledge the helpful comments of the anonymous referee who suggested important improvements to the style and layout of the paper as well as a shorter proof for Corollary 39.

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