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Locating terms in the Stern–Brocot tree

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ABSTRACT

In this paper we discover an efficient method for answering two related questions involving the Stern–Brocot tree: *What is the* jth *term in the* nth *level of the tree?* and *What is the exact position of the fraction* $\frac{r}{s}$ *in the tree?*

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1. Introduction

The Stern–Brocot tree is an infinite binary tree made up of **mediants**. A mediant is a fraction that is formed from two other fractions by simply adding their numerators to obtain a new numerator and adding their denominators to obtain a new denominator. This operation is known as **child's addition**. For example, the mediant of $\frac{3}{8}$ and $\frac{2}{5}$ is $\frac{5}{13}$. Throughout this paper the symbol $\oplus$ when applied to two fractions denotes child's addition. It can be shown (for example, Graham et al., [6]) that any mediant lies between the two fractions from which it was formed. Hughes and Niqui, [7,11], represent the tree using exact rational arithmetic techniques based on Gosper [4,5]. Graham et al., [6] represents the tree using a Right–Left binary notation. Lennerstad and Lundberg [9] use the tree as a tool for studying optimal ways of decomposing any rational number $\frac{a}{b}$ in c-decomposition ratios in order to give rational estimates of $\frac{a}{b}$ from above and below. Their technique is used to solve a general version of the 4/3 conjecture [10]. It has interesting ramifications in, for example, deciding the most equitable distribution of scarce resources over a population that has formed itself into differently-sized groups.

Our aim in this paper is to discover a simple decomposition method for identifying both the level and the position within the level of every reduced fraction in the tree by using a technique based on:

(i) a ternary radix-2 number system, and
(ii) the continued fraction of each node in the tree.

The importance of this technique lies in its ease of use and its applicability beyond the Stern–Brocot tree. For example, the techniques of this paper are applicable to the landmark paper of Calkin and Wilf [2] on what has come to be styled the Calkin–Wilf tree—a tree similar to the Stern–Brocot tree.

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Gibbons et al. [3] observe, based on the work of Newman [8], that entries in the Calkin–Wilf tree are more easily identified than entries in the Stern–Brocot tree. We hope to address this issue and other aspects of the Stern–Brocot and Calkin–Wilf trees in subsequent papers.

Throughout this paper we denote $\mathbb{N}$ as the natural numbers including 0, and $\mathbb{N}^*$ as the natural numbers excluding 0.

We now formulate the Stern–Brocot tree algebraically.

2. The Stern–Brocot sequence

Definition 1 (*Stern–Brocot sequence*). Let $A_0 = \langle a_{0,1}, a_{0,2}\rangle$ represent an ordered sequence consisting of two arbitrary integers, and define for $n \geq 1$,

$$A_n = \langle a_{n,1}, a_{n,2}, \ldots, a_{n,2^n+1}\rangle$$

as the sequence for which, for $k \geq 1$,

$$a_{n,2k-1} = a_{n-1,k} \quad \text{and} \tag{2.1}$$

$$a_{n,2k} = a_{n-1,k} + a_{n-1,k+1}. \tag{2.2}$$

Similarly let $B_0 = \langle b_{0,1}, b_{0,2}\rangle$ represent another sequence consisting of two arbitrary terms, and define for $n \geq 1$,

$$B_n = \langle b_{n,1}, b_{n,2}, \ldots, b_{n,2^n+1}\rangle$$

as another sequence for which,

$$b_{n,2k-1} = b_{n-1,k} \quad \text{and} \tag{2.3}$$

$$b_{n,2k} = b_{n-1,k} + b_{n-1,k+1}. \tag{2.4}$$

Then the sequence defined by $H_n = \langle h_{n,1}, h_{n,2}, \ldots, h_{n,2^n+1}\rangle$ where $h_{n,i} = \frac{a_{n,i}}{b_{n,i}}$ and $H_0 = \langle \frac{0}{1}, \frac{1}{0}\rangle$ is called the **Stern–Brocot sequence** of order n. It represents the ordered sequence containing both the first n generations of mediants based on H_0, and the terms of H_0 itself. We call $h_{n-1,k}$ the antecedent of $h_{n,2k-1}$; and $h_{n-1,k}$ and $h_{n-1,k+1}$ the left and right parents respectively of $h_{n,2k}$. Note that a parent of a term of H_n is either of the fractions from which the term is formed as a mediant.

Definition 2 (med H_{n-1}). For $n > 0$, if $H_{n-1} = \langle h_{n-1,1}, h_{n-1,2}, \ldots, h_{n-1,2^{n-1}+1}\rangle$ and $H_0 = \langle \frac{0}{1}, \frac{1}{0}\rangle$ represents the zeroth level of the Stern–Brocot tree, then the nth level of the Stern–Brocot tree is defined as med H_{n-1} where

$$\begin{aligned}\text{med } H_{n-1} &= \langle (h_{n-1,1} \oplus h_{n-1,2}), (h_{n-1,2} \oplus h_{n-1,3}), \ldots, (h_{n-1,2^{n-1}} \oplus h_{n-1,2^{n-1}+1})\rangle \\ &= \langle h_{n,2}, h_{n,4}, h_{n,6}, \ldots, h_{n,2^n}\rangle.\end{aligned}$$

From Definition 2, levels 0 to n of the Stern–Brocot tree represent a binary tree made up of mediants formed from adjacent terms in the Stern–Brocot sequence of order n. Lines in the tree connect each node to its closest parent. See Fig. 1. It can be readily shown (for example, Graham et al., [6]) that the Stern–Brocot tree (and sequence) contains all nonnegative reduced fractions once each. Definition 1 implicitly incorporates the operation of interleaving. We now formally define interleaving.

3. Interleaving and the Stern–Brocot tree

Definition 3 (*Interleave Operator*). We denote by #, the interleave operator acting on two ordered sets $A = \langle a_1, a_2, \ldots, a_{k+1}\rangle$ and $B = \langle b_1, b_2, \ldots, b_k\rangle$, such that

$$A\#B = \langle a_1, b_1, a_2, b_2, \ldots, b_k, a_{k+1}\rangle.$$

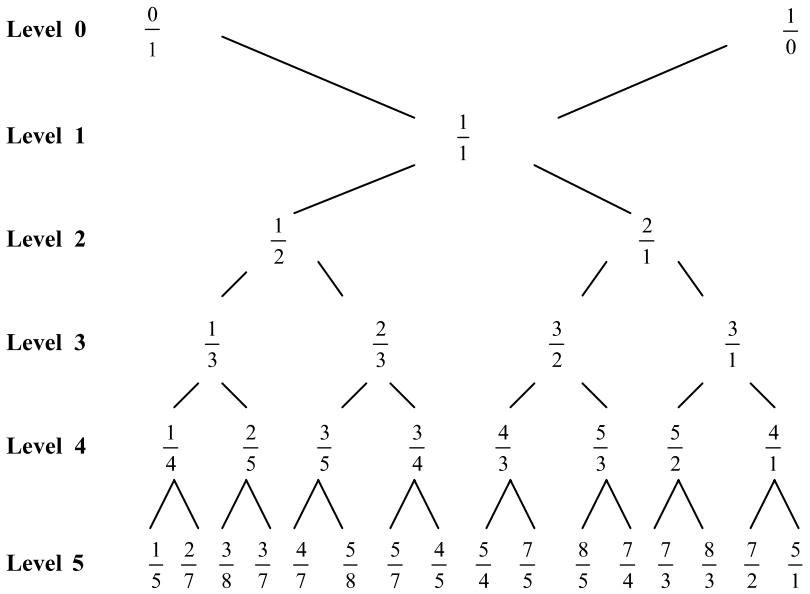


Fig. 1. Levels 0 to 5 of the Stern–Brocot tree.

In Definition 1, consecutive terms of A_{n-1} reappear as consecutive odd terms in A_n. Thus A_{n-1} interleaves with the even terms of A_n to form A_n. Similar comments exist for B_n and therefore for H_n. This gives rise to the following theorem concerning H_n based on the interleave operation in Definition 3.

Theorem 1. *Let* $H_0 = \left\langle \frac{0}{1}, \frac{1}{0} \right\rangle$. *For* $n \geq 1$,

$$H_n = H_{n-1} \# \text{med}\, H_{n-1}.$$

Proof. By Definition 1, with $H_0 = \left\langle \frac{0}{1}, \frac{1}{0} \right\rangle$,

$$h_{n,2k-1} = h_{n-1,k} \quad \text{and}$$
$$h_{n,2k} = h_{n-1,k} \oplus h_{n-1,k+1}.$$

Thus

$$\begin{aligned} H_{n-1} &= \left\langle h_{n-1,1}, h_{n-1,2}, \ldots, h_{n-1,2^{n-1}+1} \right\rangle \\ &= \left\langle h_{n,1}, h_{n,3}, \ldots, h_{n,2^n+1} \right\rangle. \end{aligned}$$

Also by Definition 2, $\text{med}\, H_{n-1} = \left\langle h_{n,2}, h_{n,4}, \ldots, h_{n,2^n} \right\rangle$.

Since from Definition 1, $H_n = \left\langle h_{n,1}, h_{n,2}, \ldots, h_{n,2^n+1} \right\rangle$ we have by Definition 3, $H_n = H_{n-1} \# \text{med}\, H_{n-1}$ where $H_0 = \left\langle \frac{0}{1}, \frac{1}{0} \right\rangle$. □

The following well-known result (For example, Graham et al., [6]) shows that each term in the Stern–Brocot tree is in its reduced form.

Theorem 2. *Let*

$$h_{n,i} = \frac{a_{n,i}}{b_{n,i}} \quad \textit{and}$$
$$h_{n,i+1} = \frac{a_{n,i+1}}{b_{n,i+1}}$$

be two consecutive terms in H_n. Then for all i, $0 < i \leq 2^n$,

$$a_{n,i+1}b_{n,i} - a_{n,i}b_{n,i+1} = 1.$$

Proof. We prove this by induction on n.
For $n = 0$, the relation is true ($1.1 - 0.0 = 1$).
Suppose, for some value m, our theorem holds.
For i odd,

$$\begin{aligned}
&a_{m+1,i+1}b_{m+1,i} - a_{m+1,i}b_{m+1,i+1}\\
&\quad = \left(a_{m,\frac{i+1}{2}} + a_{m,\frac{i+1}{2}+1}\right) b_{m,\frac{i+1}{2}} - a_{m,\frac{i+1}{2}} \left(b_{m,\frac{i+1}{2}} + b_{m,\frac{i+1}{2}+1}\right) \text{ by (2.1) to (2.4)}\\
&\quad = a_{m,\frac{i+1}{2}+1}b_{m,\frac{i+1}{2}} - a_{m,\frac{i+1}{2}}b_{m,\frac{i+1}{2}+1}\\
&\quad = 1.
\end{aligned}$$

For i even,

$$\begin{aligned}
&a_{m+1,i+1}b_{m+1,i} - a_{m+1,i}b_{m+1,i+1}\\
&\quad = a_{m,\frac{i}{2}+1}\left(b_{m,\frac{i}{2}} + b_{m,\frac{i}{2}+1}\right) - \left(a_{m,\frac{i}{2}} + a_{m,\frac{i}{2}+1}\right) b_{m,\frac{i}{2}+1} \text{by (2.1) to (2.4)}\\
&\quad = a_{m,\frac{i}{2}+1}b_{m,\frac{i}{2}} - a_{m,\frac{i}{2}}b_{m,\frac{i}{2}+1}\\
&\quad = 1. \quad \square
\end{aligned}$$

Theorem 2 indicates that $a_{n,i}$ and $b_{n,i}$ are relatively prime for all n and i. It follows that the mediant of $h_{n,i}$ and $h_{n,i+1}$ is in its reduced form.

We now show that the Stern–Brocot tree can be viewed as a set of *left and right diagonals*.

4. Left diagonals in the Stern–Brocot tree

Definition 4 (*Left Diagonals*). We define $\frac{0}{1}$ and $\frac{1}{0}$ as the zeroth and first terms respectively from the left in the zeroth level.

(i) For $k \in \mathbb{N}^*$, the kth **left diagonal** of the tree, $\mathcal{L}_k$, is the sequence made up of each kth term from each level beginning at the first level.
(ii) The **first** $^+$ **left diagonal** of the tree, $\mathcal{L}_{1^+}$, is the first left diagonal with the term $\frac{1}{0}$.

Left diagonals are read from upper right to lower left of the tree.

Example 1. $\mathcal{L}_1 = \left\langle \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \right\rangle$, $\mathcal{L}_{1^+} = \left\langle \frac{1}{0}, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots \right\rangle$, $\mathcal{L}_2 = \left\langle \frac{2}{1}, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \ldots \right\rangle$, $\mathcal{L}_3 = \left\langle \frac{3}{2}, \frac{3}{5}, \frac{3}{8}, \frac{3}{11}, \ldots \right\rangle$ and $\mathcal{L}_4 = \left\langle \frac{3}{1}, \frac{3}{4}, \frac{3}{7}, \frac{3}{10}, \ldots \right\rangle$.

Definition 5 (*Sum of Diagonals*). Let $\mathcal{L}_s = \langle m_{s,1}, m_{s,2}, m_{s,3}, \ldots \rangle$ and $\mathcal{L}_t = \langle m_{t,1}, m_{t,2}, m_{t,3}, \ldots \rangle$ represent the sth and tth left diagonals respectively. The **sum** of these two left diagonals is

$$\mathcal{L}_s \dagger \mathcal{L}_t = \langle m_{s,1} \oplus m_{t,1}, m_{s,2} \oplus m_{t,2}, m_{s,3} \oplus m_{t,3}, \ldots \rangle \tag{4.1}$$

where all terms in the right hand side of (4.1) are expressed in reduced form.

Remark 1. We note that $\dagger$ is a commutative operation, since

$$\begin{aligned}
\mathcal{L}_s \dagger \mathcal{L}_t &= \langle m_{s,1} \oplus m_{t,1}, m_{s,2} \oplus m_{t,2}, m_{s,3} \oplus m_{t,3}, \ldots \rangle\\
&= \langle m_{t,1} \oplus m_{s,1}, m_{t,2} \oplus m_{s,2}, m_{t,3} \oplus m_{s,3}, \ldots \rangle\\
&= \mathcal{L}_t \dagger \mathcal{L}_s.
\end{aligned}$$

Definition 6 (*Symmetric Complements*). Two terms found on the same level of the tree that are equidistant from either end of the level are designated **symmetric complements**.

Lemma 1. *For* $k \in \mathbb{N}^*$, $\mathcal{L}_k = \langle h_{n,2k}, h_{n+1,2k}, h_{n+2,2k}, \ldots \rangle$ *where* $n = \lceil \log_2 k \rceil + 1$.

Proof. By Definition 2, $\mathcal{L}_k = \langle h_{n,2k}, h_{n+1,2k}, h_{n+2,2k}, \ldots \rangle$ where n is the lowest level of the tree that has a kth element.

For $n > 0$, the nth level of the tree has 2^{n-1} elements. Accordingly, $2^{n-2} < k \leq 2^{n-1}$, where $k \in \mathbb{N}^*$. That is, $n = \lceil \log_2 k \rceil + 1$. □

Lemma 2. *For all* k, *where* $1 \leq k \leq 2^n + 1$,

$$a_{n,k} = b_{n,2^n-k+2}.$$

Proof. By induction on n.

We have for $n = 0$,

$$a_{0,1} = 0 = b_{0,2}$$
$$a_{0,2} = 1 = b_{0,1}$$

Suppose the lemma is true for some n. Consider the two possible cases.

(i) $k = 2l - 1$. Then by (2.1) and (2.3) and the induction hypothesis,

$$\begin{aligned} a_{n+1,2l-1} &= a_{n,l} \\ &= b_{n,2^n-l+2} \\ &= b_{n+1,2^{n+1}-2l+3}. \end{aligned}$$

(ii) $k = 2l$. Then by (2.2) and (2.4) and the induction hypothesis,

$$\begin{aligned} a_{n+1,2l} &= a_{n,l} + a_{n,l+1} \\ &= b_{n,2^n-l+2} + b_{n,2^n-l+1} \\ &= b_{n+1,2^{n+1}-2l+2}. \end{aligned}$$

And so the lemma is true for $n + 1$. □

Corollary 1. *For all* k, *where* $1 \leq k \leq 2^n + 1$,

$$h_{n,k} = \frac{1}{h_{n,2^n-k+2}}.$$

Proof. Follows immediately from Lemma 2. □

Remark 2. Corollary 1 reveals that symmetric complements are reciprocals of each other and applies both to fractions and the pseudofractions $\frac{0}{1}$ and $\frac{1}{0}$, where $\frac{0}{1}$ and $\frac{1}{0}$ are reciprocals of each other. That is, Corollary 1 is applicable to every level of the tree.

Lemma 3. *If* $k = 2^i(2j + 1)$ *where* $i, j \in \mathbb{N}$, *then*

$$h_{n,k+1} = h_{n-i,2j+2}.$$

Proof. There are two cases

(i) For k odd (that is, $i = 0$)

$$h_{n,k+1} = h_{n,2j+2}.$$

(ii) For k even (that is, $i > 0$), repeatedly using $h_{n,2p-1} = h_{n-1,p}$ yields

$$\begin{aligned} h_{n,k+1} &= h_{n-1,2^{i-1}(2j+1)+1} \\ &= h_{n-2,2^{i-2}(2j+1)+1} \\ &\vdots \\ &= h_{n-i,(2j+1)+1} \\ &= h_{n-i,2j+2}. \quad \square \end{aligned}$$

Lemma 4. *If $k = 2^i(2j+1)$ where $i, j \in \mathbb{N}$, then*

(i) $\mathcal{L}_{2k} = \begin{cases} \mathcal{L}_k \dagger \mathcal{L}_{1^+}, & j = 0 \\ \mathcal{L}_k \dagger \mathcal{L}_{j+1}, & j > 0 \end{cases}$

(ii) $\mathcal{L}_{2k+1} = \mathcal{L}_{k+1} \dagger \mathcal{L}_{j+1}$.

Proof. (i) Since $h_{n,2p} = h_{n-1,p} \oplus h_{n-1,p+1}$, we have using Lemma 3 with $k := 2k$ and for $t \geq 0$,

$$\begin{aligned} h_{n+t,4k} &= h_{n-1+t,2k} \oplus h_{n-1+t,2k+1} \\ &= h_{n-1+t,2k} \oplus h_{n-2-i+t,2j+2}. \end{aligned} \tag{4.2}$$

Now by Lemma 1, $\mathcal{L}_m = \langle h_{n,2m}, h_{n+1,2m}, h_{n+2,2m}, \ldots \rangle$ where $n = \lceil \log_2 m \rceil + 1$. That is, $2^{n-2} < m \leq 2^{n-1}$. Let $m = 2k$. Then the left hand terms of (4.2), for $t = 0, 1, 2, \ldots$ form $\mathcal{L}_{2k}$ where

$$2^{n-3} < k \leq 2^{n-2} \tag{4.3}$$

that is, $n - 1 = \lceil \log_2 k \rceil + 1$. It follows that

$$\mathcal{L}_k = \langle h_{n-1,2k}, h_{n,2k}, h_{n+1,2k}, \ldots \rangle$$

which is also the set of first terms on the right hand side of (4.2) for $t = 0, 1, 2, \ldots$

Consider the two cases for j:

(a) $j = 0$.

For $j = 0$, $k = 2^i$. From (4.3), $i = n - 2$, and the second term on the right hand side of (4.2) becomes $h_{t,2}$. Since $\mathcal{L}_{1^+} = \langle h_{0,2}, h_{1,2}, h_{2,2}, \ldots \rangle$ the set of second terms on the right hand side of (4.2) for $t = 0, 1, 2, \ldots$ is $\mathcal{L}_{1^+}$. Accordingly, for $j = 0$,

$$\mathcal{L}_{2k} = \mathcal{L}_k \dagger \mathcal{L}_{1^+}.$$

(b) $j > 0$.

Since $2^{n-3} < k \leq 2^{n-2}$,

$$\begin{aligned} &2^{n-3-i} < 2j + 1 \leq 2^{n-2-i} \\ &2^{n-4-i} < j + \frac{1}{2} \leq 2^{n-3-i} \\ &\text{that is, } 2^{n-4-i} < j + 1 \leq 2^{n-3-i}. \end{aligned}$$

It follows from Lemma 1 that $\mathcal{L}_{j+1} = \langle h_{n-2-i,2j+2}, h_{n-1-i,2j+2}, h_{n-i,2j+2}, \ldots \rangle$ which for $t = 0, 1, 2, \ldots$ represents the set of second terms on the right hand side of (4.2). Accordingly, for $j > 0$, $\mathcal{L}_{2k} = \mathcal{L}_k \dagger \mathcal{L}_{j+1}$.

(ii) Since $h_{n,2p} = h_{n-1,p} \oplus h_{n-1,p+1}$, we have using Lemma 3 with $k := 2k$ and for $t \geq 0$,

$$\begin{aligned} h_{n+t,4k+2} &= h_{n-1+t,2k+1} \oplus h_{n-1+t,2k+2} \\ &= h_{n-2-i+t,2j+2} \oplus h_{n-1+t,2k+2}. \end{aligned} \tag{4.4}$$

By Lemma 1, the set of first terms in the left hand side of (4.4) is $\mathcal{L}_{2k+1}$, where $n = \lceil \log_2 (2k+1) \rceil + 1$.

Again by Lemma 1, $\mathcal{L}_{j+1} = \langle h_{n-2-i,2j+2}, h_{n-1-i,2j+2}, h_{n-i,2j+2}, \ldots \rangle$ where $n-2-i = \lceil \log_2 (j+1) \rceil + 1$ represents the set of first terms in the right hand side of (4.4).

Since $k > 0$ and $2^{n-2} < 2k+1 \leq 2^{n-1}$,

$$2^{n-3} < k + \frac{1}{2} < k + 1 \leq 2^{n-2}.$$

It follows that $\mathcal{L}_{k+1} = \langle h_{n-1,2k+2}, h_{n,2k+2}, h_{n+1,2k+2}, \ldots \rangle$ which for $t = 0, 1, 2, \ldots$ represents the set of second terms in the right hand side of (4.4), where by Lemma 1, $n - 1 = \lceil \log_2 (k+1) \rceil + 1$.

Accordingly, by Remark 1, $\mathcal{L}_{2k+1} = \mathcal{L}_{j+1} \dagger \mathcal{L}_{k+1} = \mathcal{L}_{k+1} \dagger \mathcal{L}_{j+1}$. □

As a result of this lemma, we can obtain any $\mathcal{L}_k$ from $\mathcal{L}_i$ for values of $i < k$.

Example 2. The following are left diagonals shown as the sum of two left diagonals.

$$\mathcal{L}_2 = \mathcal{L}_1 \dagger \mathcal{L}_{1^+} = \mathcal{L}_1 \dagger \mathcal{L}_4$$
$$\mathcal{L}_3 = \mathcal{L}_1 \dagger \mathcal{L}_2 = \mathcal{L}_1 \dagger \mathcal{L}_6$$
$$\mathcal{L}_4 = \mathcal{L}_2 \dagger \mathcal{L}_{1^+} = \mathcal{L}_2 \dagger \mathcal{L}_8.$$

Each example contains two sets of summands to show that each left diagonal does not have a unique set of summands. However, each first set of summands begins on earlier levels of the tree to its respective sum, as given in Lemma 4.

Theorem 3 (*The "$\frac{a}{a+b}$ Rule" for Left Diagonals*). *For $n > 1$, $k > 0$, let $\frac{a}{b}$ be $h_{n-1,2k}$, the kth entry in the $(n-1)$th level of the Stern–Brocot tree. Then*

(i) *$h_{n,2k}$, the kth entry in the nth level, is $\frac{a}{a+b}$.*
(ii) *$h_{n,2^n-2k+2}$, the $\left(2^{n-1} - k + 1\right)$th entry in the nth level, is $\frac{a+b}{a}$.*

Proof. (i) Since $\mathcal{L}_1 = \left\langle \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots \right\rangle$ and $\mathcal{L}_{1^+} = \left\langle \frac{1}{0}, \frac{1}{1}, \frac{1}{2}, \ldots \right\rangle$, each term $\frac{a}{b}$ in $\mathcal{L}_1$ and $\mathcal{L}_{1^+}$ is followed by the term $\frac{a}{a+b}$. We call this pattern the '$\frac{a}{a+b}$ rule'. Now the sum of any two left diagonals that abide by the '$\frac{a}{a+b}$ rule' must itself also abide by the '$\frac{a}{a+b}$ rule'. To demonstrate that this is so, let

$$\mathcal{L} = \left\langle \frac{e}{f}, \frac{e}{e+f}, \frac{e}{2e+f}, \ldots \right\rangle \quad \text{and}$$
$$\mathcal{L}' = \left\langle \frac{c}{d}, \frac{c}{c+d}, \frac{c}{2c+d}, \ldots \right\rangle$$

be the two summands of the sum. Then

$$\mathcal{L} \dagger \mathcal{L}' = \left\langle \frac{e+c}{f+d}, \frac{e+c}{(e+c)+(f+d)}, \frac{e+c}{2\,(e+c)+(f+d)}, \ldots \right\rangle$$

which abides by the '$\frac{a}{a+b}$ rule'. Since by Lemma 4 all left diagonals are the sum of earlier left diagonals and $\mathcal{L}_1$ and $\mathcal{L}_{1^+}$ abide by the '$\frac{a}{a+b}$ rule', it follows that all left diagonals abide by the '$\frac{a}{a+b}$ rule'. Hence if $\frac{a}{b} = h_{n-1,2k}$, the next term in the $\mathcal{L}_k$ is $h_{n,2k} = \frac{a}{a+b}$.

(ii) Since $0 < 2^{n-1} - k + 1$ this follows from Corollary 1. □

We now show that the Stern–Brocot tree can be alternatively represented as a set of *right diagonals.*

5. Right diagonals in the Stern–Brocot tree

Definition 7 (*Right Diagonals*). We define $\frac{1}{0}$ and $\frac{0}{1}$ as the zeroth and first terms respectively from the right in the zeroth level.

(i) For $k \in \mathbb{N}^*$, the kth **right diagonal** of the tree, $\mathcal{R}_k$, is the sequence made up of each kth term taken from the end of each level beginning at the first level.
(ii) The **first$^+$ right diagonal** of the tree, $\mathcal{R}_{1^+}$, is the first right diagonal with the term $\frac{0}{1}$.

Right diagonals are read from upper left to lower right of the tree.

From Definition 7, each element of $\mathcal{L}_i$ is the symmetric complement of its corresponding element in $\mathcal{R}_i$, where $i = 1, 1^+, 2, 3, 4, \ldots$. That is, $\mathcal{R}_i$ is formed through taking reciprocals of each element of $\mathcal{L}_i$. In particular, by Lemma 1 and Corollary 1, for $k \in \mathbb{N}^*$,

$$\begin{aligned}\mathcal{R}_k &= \left\langle \frac{1}{h_{n,2k}}, \frac{1}{h_{n+1,2k}}, \frac{1}{h_{n+2,2k}}, \ldots \right\rangle \\ &= \left\langle h_{n,2^n-2k+2}, h_{n+1,2^n-2k+2}, h_{n+2,2^n-2k+2}, \ldots \right\rangle \end{aligned} \tag{5.1}$$

where $n = \lceil \log_2 k \rceil + 1$.

Example 3. $\mathcal{R}_1 = \left\langle \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \ldots \right\rangle$, $\mathcal{R}_{1^+} = \left\langle \frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \ldots \right\rangle$, $\mathcal{R}_2 = \left\langle \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \right\rangle$, $\mathcal{R}_3 = \left\langle \frac{2}{3}, \frac{5}{3}, \frac{8}{3}, \ldots \right\rangle$ and $\mathcal{R}_4 = \left\langle \frac{1}{3}, \frac{4}{3}, \frac{7}{3}, \ldots \right\rangle$.

Theorem 4 (*The "$\frac{a+b}{b}$ Rule" for Right Diagonals*). *For $n > 1$, let $\frac{a}{b}$ be the kth entry from the rightmost end of the $(n-1)$th level of the Stern–Brocot tree. Then*

(i) *The kth entry from the rightmost end of the nth level is $\frac{a+b}{b}$.*
(ii) *The $\left(2^{n-1} - k + 1\right)$th entry from the rightmost end of the nth level is $\frac{b}{a+b}$.*

Equivalently,

Theorem 5. *For $n > 1$, let $\frac{a}{b}$ be the kth entry of the $(n-1)$th level of the Stern–Brocot tree. Then*

(i) *The $\left(k + 2^{n-2}\right)$th entry in the nth level is $\frac{a+b}{b}$.*
(ii) *The $\left(2^{n-2} - k + 1\right)$th entry of the nth level is $\frac{b}{a+b}$.*

Proof. (i) Let $\frac{a}{b}$ be the kth entry of the $(n-1)$th level of the Stern–Brocot tree. By Remark 2, $\frac{b}{a}$ is the $\left(2^{n-2} - k + 1\right)$th of the $(n-1)$th level. By Theorem 3 (ii), $\frac{b+a}{b}$ is the $\left(2^{n-2} + k\right)$th entry in the nth level.

(ii) Remark 2 applied to (i) gives the result in (ii). □

Our interest now lies in representing left and right diagonals in their continued fraction form.

6. Continued fraction form of left and right diagonals

We employ the notation $\{a_0; a_1, a_2, \ldots, a_n\}$ to represent the continued fraction expansion

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}.$$

Note that if $a_i = 0$,

$$\{a_0; a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n\} = \{a_0; a_1, \ldots, a_{i-2}, a_{i-1} + a_{i+1}, a_{i+2}, \ldots, a_n\}.$$

Theorem 6. *If $\{a_0; a_1, a_2, \ldots, a_n\}$ is the first term in a right diagonal of the Stern–Brocot tree, then the remaining terms are successively: $\{a_0 + t; a_1, \ldots, a_n\}$ for $t = 1, 2, \ldots$.*

Proof. If $\frac{c}{d} = \{b_0; b_1, b_2, \ldots, b_n\}$ is any term in a right diagonal, the next term is by Theorem 4,

$$\frac{c+d}{d} = 1 + \frac{c}{d} = \{b_0 + 1; b_1, b_2, \ldots, b_n\}. \quad \square$$

Theorem 7. *If* $\{a_0; a_1, a_2, \ldots, a_n\}$ *is the first term in a left diagonal of the Stern–Brocot tree, then the remaining terms are successively:* $\{0; t, a_0, \ldots, a_n\}$ *for* $t = 1, 2, \ldots$.

Proof. If $\frac{c}{d} = \{b_0; b_1, \ldots, b_n\}$ is the first term of $\mathcal{L}_k$, $k \geq 1$, then $\frac{d}{c} = \{0; b_0, b_1, \ldots, b_n\}$ is the first term of $\mathcal{R}_k$. By Theorem 6, the terms of $\mathcal{R}_k$ are $\{t; b_0, b_1, \ldots, b_n\}$, $t = 0, 1, 2, \ldots$ where successive terms are ordered on t. Thus by (5.1), the terms of $\mathcal{L}_k$ are $\{0; t, b_0, \ldots, b_n\}$, $t = 0, 1, 2, \ldots$ where successive terms are ordered on t. □

7. Additive and normalised additive factorisation

We are now closer to answering our initial question: "What is the jth term in the nth level of the tree?" To answer this question we will use our knowledge of left and right diagonals. But first we introduce the concept of additive factorisation.

Definition 8 (*Additive Factors*). Let $\lambda_0, \lambda_1, \ldots, \lambda_k$ be the additive factors of j defined through the recurrence

$$\lambda_i = \lceil \log_2 j_i \rceil$$

where

$$j_i = \begin{cases} j, & \text{for } i = 0 \\ 2^{\lambda_{i-1}} - j_{i-1} & \text{for } i = 1, 2, \ldots, k \end{cases}$$

and k is the smallest value of i for which $\log_2 j_i = \lceil \log_2 j_i \rceil$. That is, $j_k = 2^{\lambda_k}$.

Lemma 5. *If* $\lambda_0, \lambda_1, \ldots, \lambda_k$ *are the additive factors of* j *then*

(i) *for* $k > 0$, $\lambda_0 > \lambda_1 > \cdots > \lambda_{k-1} > \lambda_k + 1$ *and*
(ii) $j = \sum_{i=0}^{k} (-1)^i 2^{\lambda_i}$.

Proof. (i) For each i, $0 \leq i < k$,

$$2^{\lambda_i - 1} < j_i < 2^{\lambda_i}. \tag{7.1}$$

That is, $-2^{\lambda_i} < -j_i < -2^{\lambda_i - 1}$. Therefore

$$0 < 2^{\lambda_i} - j_i = j_{i+1} < 2^{\lambda_i - 1}. \tag{7.2}$$

Since by Definition 8, $\lambda_{i+1} = \lceil \log_2 j_{i+1} \rceil$, combining (7.1) and (7.2) yields

$$j_{i+1} < 2^{\lambda_{i+1}} \leq 2^{\lambda_i - 1} < j_i < 2^{\lambda_i}.$$

Accordingly, $j_0 > j_1 > \cdots > j_k$ and

$$\lambda_0 > \lambda_1 > \cdots > \lambda_k. \tag{7.3}$$

Suppose $\lambda_{k-1} = \lambda_k + 1$, then $2^{\lambda_{k-1}} = 2^{\lambda_k + 1}$.

From Definition 8, $j_k = 2^{\lambda_{k-1}} - j_{k-1} = 2^{\lambda_k}$. That is, $j_{k-1} = 2^{\lambda_{k-1}} - 2^{\lambda_k} = 2^{\lambda_k + 1} - 2^{\lambda_k} = 2^{\lambda_k} = j_k$, an impossibility. Thus $\lambda_{k-1} \neq \lambda_k + 1$. From (7.3) the result follows.

(ii) From Definition 8, $j_1 = 2^{\lambda_0} - j_0$. Therefore

$$\begin{aligned} j = j_0 &= 2^{\lambda_0} - 2^{\lambda_1} + j_2 \\ &= 2^{\lambda_0} - 2^{\lambda_1} + 2^{\lambda_2} - j_3 \\ &\;\;\vdots \\ &= 2^{\lambda_0} - 2^{\lambda_1} + 2^{\lambda_2} - \cdots + (-1)^k 2^{\lambda_k} \\ &= \sum_{i=0}^{k} (-1)^i 2^{\lambda_i}. \quad \square \end{aligned}$$

Definition 9 (*Normalised Additive Factors*). If $\lambda_0, \lambda_1, \ldots, \lambda_k$ are the additive factors of j, then the normalised additive factors of j are

(i) For k even, $\lambda_0, \lambda_1, \ldots, \lambda_k$;
(ii) For k odd, $\lambda_0, \lambda_1, \ldots, \lambda_{k-1}, \lambda_k + 1, \lambda_k$.

The following theorem guarantees that we can represent any integer, j, as a ternary radix-2 number system consisting of an odd number of terms; and that consecutive powers in this power series are the consecutive normalised additive factors of j.

Theorem 8. *If $b_0, b_1, \ldots, b_s$ are the normalised additive factors of j, then*

(i) *s is even; and*
(ii) $j = \sum_{i=0}^{s} (-1)^i 2^{b_i}$.

Proof. (i) This follows from Definition 9.
(ii) Let $\lambda_0, \lambda_1, \ldots, \lambda_k$ be the additive factors of j. Then
(a) For k even, by Definition 9, $s = k$ and the normalised additive factors of j are the additive factors of j. The result follows by Lemma 5.
(b) For k odd, by Definition 9, $s = k + 1$ and

$$\begin{aligned}\sum_{i=0}^{s} (-1)^i 2^{b_i} &= \sum_{i=0}^{k-1} (-1)^i 2^{\lambda_i} - 2^{(\lambda_k+1)} + 2^{\lambda_k} \\ &= \sum_{i=0}^{k-1} (-1)^i 2^{\lambda_i} - 2^{\lambda_k} \\ &= \sum_{i=0}^{k} (-1)^i 2^{\lambda_i} \\ &= j. \quad \square\end{aligned}$$

Example 4. The number 6 has additive factors 3, 1 and normalised additive factors 3, 2, 1. Accordingly, it is additively factorised as $6 = 2^3 - 2^1$and normally additively factorised as $6 = 2^3 - 2^2 + 2^1$.

8. Identifying terms within the Stern–Brocot tree

Using our normalised additive factors of j, we can now answer the first of our questions: *What is the jth term in the nth level of the tree?*

Lemma 6. *Let the normalised additive factors of j be $b_{j,0}, b_{j,1}, \ldots, b_{j,k}$.*
Then if $2^{n-1} < j \leq 2^n$, the normalised additive factors of $2^n - j + 1$ are

(i) *For $j = 2^n$: 0*
(ii) *For $2^{n-1} < j < 2^n$:*
(a) *For $b_{j,k} > 0$: $b_{j,1}, b_{j,2}, \ldots, b_{j,k}, 0$*
(b) *For $b_{j,k} = 0$: $b_{j,1}, b_{j,2}, \ldots, b_{j,k-1}$.*

Proof. (i) $2^n - j + 1 = 1$. The result follows from Definition 9.
(ii) We have $j = 2^{b_{j,0}} - 2^{b_{j,1}} + 2^{b_{j,2}} - \cdots + 2^{b_{j,k}}$. Since $2^{n-1} < j < 2^n$, $b_{j,0} = n$, and so, $j = 2^n - 2^{b_{j,1}} + 2^{b_{j,2}} - \cdots + 2^{b_{j,k}}$. Therefore

$$\begin{aligned}2^n - j + 1 &= 2^n - \left(2^n - 2^{b_{j,1}} + 2^{b_{j,2}} - \cdots + 2^{b_{j,k}}\right) + 1 \\ &= \begin{cases} 2^{b_{j,1}} - 2^{b_{j,2}} + \cdots - 2^{b_{j,k}} + 2^0, & \text{if } b_{j,k} > 0 \\ 2^{b_{j,1}} - 2^{b_{j,2}} + \cdots + 2^{b_{j,k-1}}, & \text{if } b_{j,k} = 0. \end{cases} \quad \square\end{aligned}$$

Remark 3. If we allowed the last two additive factors of a number to be zero, it is clear that a j with additive factors $b_{j,0}, b_{j,1}, b_{j,2}, \ldots, b_{j,k-1}$where $b_{j,k-1} > 0$ also has additive factors $b_{j,0}, b_{j,1}, b_{j,2}, \ldots, b_{j,k-1}, 0, 0$.

Hence, if we allow this, Lemma 6 could be alternatively stated as:

If the normalised additive factors of j are $b_{j,0}, b_{j,1}, \ldots, b_{j,k}$ and if $2^{n-1} < j \leq 2^n$, then the normalised additive factors of $2^n - j + 1$ are

$$b_{j,1}, b_{j,2}, \ldots, b_{j,k}, 0.$$

Example 5. $7 = 2^3 - 2^1 + 2^0$ has normalised additive factors 3,1,0.

$2^3 - 7 + 1 = 2$ has normalised additive factor 1. Under the relaxed definition in Remark 3, 2 could also be said to have normalised additive factors 1,0,0.

We are now able to state the first of our main results.

Theorem 9. *Let the normalised additive factors of j be $b_{j,0}, b_{j,1}, \ldots, b_{j,k}$. Then the jth term in the nth level of the tree, for $n > 0$, is given by the continued fraction:*

$$\left\{0; n - b_{j,0} - 1, \left(b_{j,0} - b_{j,1}\right), \left(b_{j,1} - b_{j,2}\right), \ldots, \left(b_{j,k-1} - b_{j,k}\right), b_{j,k} + 1\right\}. \tag{8.1}$$

Proof. (i) We show that the theorem is true for $j = 1$, from level 1 onwards.

The theorem is true in level 1 since j can only be 1 and has only one normalised additive factor, namely, $b_{1,0} = 0$, and $\left\{0; n - b_{1,0} - 1, b_{1,0} + 1\right\} = \{0; 0, 1\} = \frac{1}{1}$.

By Theorem 7, the theorem is true for all terms in $\mathcal{L}_1$.

(ii) We show that if the theorem is true for some jth term, it is true for the symmetric complement of the jth term.

Let the theorem be true for some j in the left half of level n, for $n > 1$. Thus $1 \leq j \leq 2^{n-2}$. Let the term at position l be the symmetric complement of the term at position j. Thus $l = 2^{n-1} - j + 1$ and $2^{n-2} < l \leq 2^{n-1}$. Let the normalised additive factors of l be $b_{l,0}, b_{l,1}, \ldots, b_{l,m}$.

From Remark 2, the lth term in the nth level of the tree is the reciprocal of the jth term in the nth level of the tree. Thus the continued fraction of the lth term in the nth level of the tree is:

$$\begin{aligned}
&\left\{n - b_{j,0} - 1; \left(b_{j,0} - b_{j,1}\right), \left(b_{j,1} - b_{j,2}\right), \ldots, \left(b_{j,k-1} - b_{j,k}\right), b_{j,k} + 1\right\} \\
&= \left\{n - b_{l,1} - 1; \left(b_{l,1} - b_{l,2}\right), \left(b_{l,2} - b_{l,3}\right), \ldots, \left(b_{l,m} - 0\right), 0 + 1\right\} \text{ by Lemma 6 and} \\
&\quad \text{Remark 3 applied to } l. \\
&= \left\{\left(b_{l,0} - b_{l,1}\right); \left(b_{l,1} - b_{l,2}\right), \left(b_{l,2} - b_{l,3}\right), \ldots, \left(b_{l,m} + 1\right)\right\} \text{ since } b_{l,0} = n - 1 \\
&= \left\{0; 0, \left(b_{l,0} - b_{l,1}\right); \left(b_{l,1} - b_{l,2}\right), \left(b_{l,2} - b_{l,3}\right), \ldots, \left(b_{l,m} + 1\right)\right\} \\
&= \left\{0; n - b_{l,0} - 1, \left(b_{l,0} - b_{l,1}\right); \left(b_{l,1} - b_{l,2}\right), \left(b_{l,2} - b_{l,3}\right), \ldots, \left(b_{l,m} + 1\right)\right\}
\end{aligned}$$

thereby establishing that if the theorem is true for some j it is true for the symmetric complement of j.

(iii) We show that (i) and (ii) imply that our theorem is true for all terms in the tree beyond the zeroth level.

From (i) our theorem is true for $\mathcal{L}_1$. From (ii) our theorem must also be true for $\mathcal{R}_1$. Thus our theorem is established for at least levels 1 and 2. By Theorem 7, our theorem is true for $\mathcal{L}_2$. Thus it is true for the left half of level 3. By (ii) again, it follows that it is true for all of level 3. By Theorem 7 again, our theorem is true for $\mathcal{L}_3$ and $\mathcal{L}_4$, and so for the left half of level 4. Proceeding in this way, our theorem is true for all levels except the zeroth level. □

Remark 4. Since for $2^{n-2} < j \leq 2^{n-1}$, $n - b_{j,0} - 1 = 0$ in the nth level, we could alternatively represent Theorem 9 as:

Let the normalised additive factors of j be $b_{j,0}, b_{j,1}, \ldots, b_{j,k}$. Then

(i) For $j \leq 2^{n-2}$, the jth term in the nth level of the tree is given by

$$\left\{0; n - b_{j,0} - 1, \left(b_{j,0} - b_{j,1}\right), \left(b_{j,1} - b_{j,2}\right), \ldots, \left(b_{j,k-1} - b_{j,k}\right), b_{j,k} + 1\right\}.$$

(ii) For $j > 2^{n-2}$, the jth term in the nth level of the tree is given by

$$\left\{\left(b_{j,0} - b_{j,1}\right); \left(b_{j,1} - b_{j,2}\right), \ldots, \left(b_{j,k-1} - b_{j,k}\right), b_{j,k} + 1\right\}.$$

Example 6. Find the 52nd term in the 8th level of the tree.
The normalised additive factors of 52 are 6, 4, 2.
By Theorem 9, the 52nd term in the 8th level of the tree is $\{0; 1, 2, 2, 3\} = \frac{17}{24}$.

Example 7. Find the 77th term in the 8th level of the tree.
Since $\lceil \log_2 77 \rceil = 7$, the 77th term is in the right half of the 8th level of the tree
The normalised additive factors of 77 are 7, 6, 4, 2, 0.
By Remark 4, the 77th term in the 8th level of the tree is

$$\{1; 2, 2, 2, 1\} = \{1; 2, 2, 3\} = \frac{24}{17}.$$

Note that the 77th term in the 8th level of the tree is the symmetric complement of the 52nd term in the 8th level of the tree of Example 6.

We are now able to develop an simple mechanism for answering our second question: *Where is the exact position of the fraction $\frac{r}{s}$ in the Stern–Brocot tree?*

Theorem 10. *If $\alpha = \{a_0; a_1, \ldots, a_k\}$ where k is even, and j represents the position in level $\sum_{i=0}^{k} a_i$ that α occupies, then*

(i) *If for $a_0 > 0$, the normalised additive factorisation of j is $j = \sum_{i=0}^{m} (-1)^i 2^{b_{j,i}}$ then $k = m$, and*

$$b_{j,i} = \left(\sum_{h=i}^{k} a_h\right) - 1.$$

(ii) *If for $a_0 = 0$, the normalised additive factorisation of j is $j = \sum_{i=0}^{m} (-1)^i 2^{b_{j,i}}$ then $k = m + 2$, and*

$$b_{j,i} = \left(\sum_{h=i+2}^{k} a_h\right) - 1.$$

Proof. We and others have shown (For example Bates et al. [1]) that α is found in level $\sum_{h=0}^{k} a_h$. Since j is the position that α occupies in level $n = \sum_{h=0}^{k} a_h$, from Remark 4,

(i) For $a_0 > 0$, j must be found in the second half of level n. That is, $j > 2^{n-2}$. Hence

$$\alpha = \left\{\left(b_{j,0} - b_{j,1}\right); \left(b_{j,1} - b_{j,2}\right), \ldots, \left(b_{j,m-1} - b_{j,m}\right), b_{j,m} + 1\right\}$$

where m is even by Theorem 8 and $b_{j,0}, b_{j,1}, \ldots, b_{j,m}$ are the normalised additive factors of j.

But since $\alpha = \{a_0; a_1, \ldots, a_k\}$ where k is even (that is, α is in either its short or long form to ensure that k is even), it follows that $m = k$ and

$$\begin{aligned}
b_{j,0} &= b_{j,1} + a_0 \\
&= b_{j,2} + a_1 + a_0 \\
&\vdots \\
&= b_{j,k} + (a_{k-1} + a_{k-2} + \cdots + a_1 + a_0) \\
&= a_k - 1 + (a_{k-1} + a_{k-2} + \cdots + a_1 + a_0) \\
&= \left(\sum_{h=0}^{k} a_h\right) - 1.
\end{aligned}$$

Also

$$\begin{aligned} b_{j,1} &= b_{j,0} - a_0 \\ &= \left(\sum_{h=1}^{k} a_h\right) - 1 \end{aligned}$$

and so on for all $b_{j,i}$ such that, in general,

$$b_{j,i} = \left(\sum_{h=i}^{k} a_h\right) - 1 \tag{8.2}$$

for $i = 0, 1, 2, \ldots$.

(ii) For $a_0 = 0$, j must be found in the first half of level n. That is, $j \leq 2^{n-2}$. Hence

$$\begin{aligned} \alpha &= \{0; n - b_{j,0} - 1, \left(b_{j,0} - b_{j,1}\right), \left(b_{j,1} - b_{j,2}\right), \\ &\quad \ldots, \left(b_{j,m-1} - b_{j,m}\right), b_{j,m} + 1\} \qquad (8.3) \\ &= \{0; a_1, \ldots, a_k\} \qquad (8.4) \end{aligned}$$

where m is even and as before, $b_{j,0}, b_{j,1}, \ldots, b_{j,m}$ are the normalised additive factors of j. Since both m and k are even, and there are $m + 3$ terms in (8.3) and $k + 1$ terms in (8.4), it follows that $m = k - 2$. Equating terms in each continued fraction expansion in (8.3) yields

$$b_{j,0} = n - a_1 - 1 = \left(\sum_{h=0}^{k} a_h\right) - a_1 - 1 = \left(\sum_{h=2}^{k} a_h\right) - 1.$$

$$b_{j,1} = b_{j,0} - a_2 = \left(\sum_{h=3}^{k} a_h\right) - 1.$$

$$\vdots$$

$$b_{j,m} = a_k - 1.$$

That is, for $i = 0, 1, 2, \ldots$,

$$b_{j,i} = \left(\sum_{h=i+2}^{k} a_h\right) - 1. \quad \square$$

We note that a continued fraction $\{a_0; a_1, \ldots, a_k\}$ can always be represented with k even, since for k odd, $\{a_0; a_1, \ldots, a_k\} = \{a_0; a_1, \ldots, a_k - 1, 1\}$.

Example 8. Where is $\frac{17}{24}$ positioned in the Stern–Brocot tree?

Answer: $\frac{17}{24} = \{0; 1, 2, 2, 3\}$. And so

$$b_0 = a_2 + a_3 + a_4 - 1 = 2 + 2 + 3 - 1 = 6$$
$$b_1 = a_3 + a_4 - 1 = 2 + 3 - 1 = 4$$
$$b_2 = a_4 - 1 = 3 - 1 = 2.$$

Thus

$$j = 2^6 - 2^4 + 2^2 = 52.$$

Also

$$n = a_0 + a_1 + a_2 + a_3 + a_4 = 0 + 1 + 2 + 2 + 3 = 8.$$

Hence $\frac{17}{24}$ represents the 52nd entry in level 8.

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