European Journal of Combinatorics 31 (2010) 1637–1661

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Linking the Calkin–Wilf and Stern–Brocot trees

Bruce Bates, Martin Bunder, Keith Tognetti

Centre for Pure Mathematics, School of Mathematics and Applied Statistics, University of Wollongong, Wollongong, NSW, 2522, Australia

ARTICLE INFO

Article history:
Received 28 November 2009
Accepted 6 April 2010
Available online 18 May 2010

ABSTRACT

Links between the Calkin–Wilf tree and the Stern–Brocot tree are discussed answering the questions: *What is the j*th *vertex in the n*th *level of the Calkin–Wilf tree?* and *Where is the vertex* $\frac{r}{s}$ *located in the Calkin–Wilf tree?* A simple mechanism is described for converting the *j*th vertex in the *n*th level of the Calkin–Wilf tree into the *j*th entry in the *n*th level of the Stern–Brocot tree. We also provide a simple method for evaluating terms in the Hyperbinary sequence thus answering a challenge raised in *Quantum* in September 1997. We also examine successors and predecessors in both trees.

© 2010 Elsevier Ltd. All rights reserved.

1. Introduction

In this paper we explore the connections between the Calkin–Wilf tree and the Stern–Brocot tree using continued fractions and a factorisation technique called normalised additive factorisation. These tools have already been used with the Stern–Brocot tree to readily identify the location of entries in that tree (see Bates et al. [2]) and can be similarly applied to the Calkin–Wilf tree to identify its entries.

The Calkin–Wilf tree has traditionally received less attention than its more familiar cousin, the Stern–Brocot tree. In recent years there have been attempts to unravel its properties and its linkages. Notable among these is of course Calkin and Wilf [5] who describe key aspects of the Calkin–Wilf tree including the fact that every reduced positive rational appears in the Calkin–Wilf tree. Their key emphasis is with the Calkin–Wilf sequence. Courtright and Sellers [6] develop a generalisation of hyperbinary partitions which they call hyper *m*-ary partitions. Klavžar et al. [9] link Stern polynomials to hyperbinary representations of $n-1$ and the Gray Code. Hinz et al. [8], in a study of the Tower of Hanoi graphs, established a linkage between the hyperbinary representation of n and Sierpiński graphs.

In this paper, we describe a mechanism for locating and identifying terms in the Calkin–Wilf tree. We also demonstrate that the Stern–Brocot tree can be constructed from the Calkin–Wilf tree, and vice versa, through a simple four-step algorithm.

E-mail addresses: bbates@uow.edu.au, Bruce_P_BATES@rta.nsw.gov.au (B. Bates), mbunder@uow.edu.au (M. Bunder), tognetti@uow.edu.au (K. Tognetti).

0195-6698/$ – see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ejc.2010.04.002

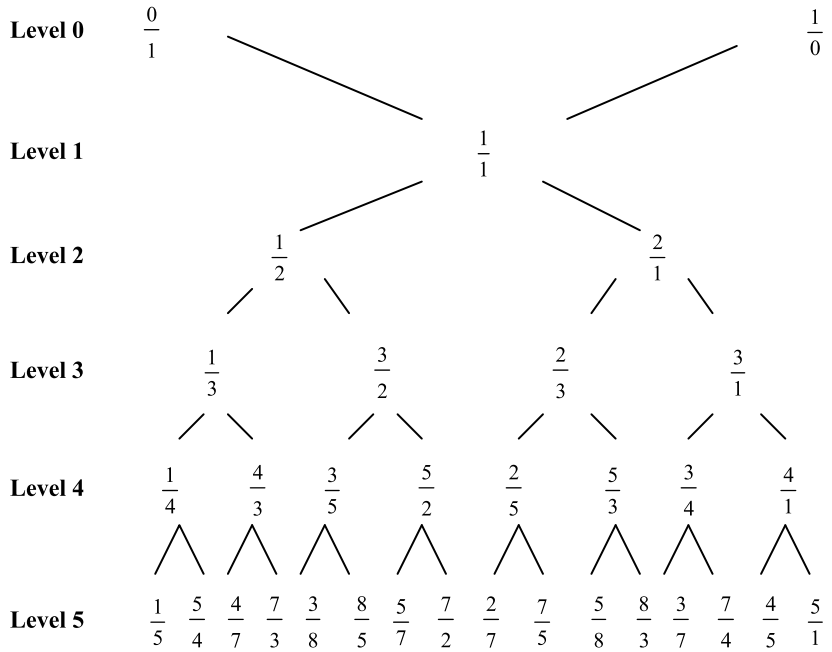


Fig. 1. Levels 0–5 of the Calkin–Wilf tree.

2. The Calkin–Wilf tree and Calkin–Wilf sequence

The following definition of the Calkin–Wilf tree is an extension of that found in [5].

Definition 1 (*Calkin–Wilf Tree*)**.** The Calkin–Wilf tree is a binary tree of fractions (called vertices) with the pseudofractions $\frac{0}{1}$ and $\frac{1}{0}$ constituting level 0, and $\frac{1}{1}$ constituting level 1. Each vertex $\frac{a}{b}$ from level 1 onwards is a parent to two children: a left child $\frac{a}{a+b}$ and a right child $\frac{a+b}{b}$. The children of a vertex are found in the next level of the tree to the vertex. The fraction $\frac{1}{1}$ is defined as the right child of $\frac{0}{1}$ and the left child of $\frac{1}{0}$.

Fig. 1 shows levels 0–5 of the Calkin–Wilf tree.

Theorem 1. *The jth vertex in level n of the Calkin–Wilf tree is the reciprocal of the jth vertex from the end of level n.*

Proof. We prove this by induction on n.

The result is true for the first two levels of the tree.

Suppose our result is true up to and including level k.

Let $\frac{a}{b}$ be the jth vertex in level k. By our supposition, $\frac{b}{a}$ is the jth vertex from the end of level k.

From Definition 1, the left child of $\frac{a}{b}$ is $\frac{a}{a+b}$ and the right child of $\frac{b}{a}$ is $\frac{a+b}{a}$. Thus the $(2j-1)$th vertex in level $k+1$ is the reciprocal of the $(2j-1)$th vertex from the end of level $k+1$. Similarly, the right child of $\frac{a}{b}$ is $\frac{a+b}{b}$ and the left child of $\frac{b}{a}$ is $\frac{b}{a+b}$. Thus the $2j$th vertex in level $k+1$ is the reciprocal of the $2j$th vertex from the end of level $k+1$. This reciprocation is true for all j in level k, and so for all entries in level $k+1$.

Hence our result is true for all n. □

As we shall demonstrate in a later section, the Calkin–Wilf tree is related to the concept of a hyperbinary expansion and the hyperbinary sequence. These we now define.

Definition 2 (*Hyperbinary Expansion*)**.** The expansion of a number as a sum of powers of 2, each power being used at most twice, is called a hyperbinary expansion.

Definition 3 (*Hyperbinary Sequence And Calkin–Wilf Sequence*)**.** Let $f(n)$ be the number of hyperbinary expansions of n with $f(0) = 1$. Then the ordered set

$$\{f(n)\} = \{1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, \ldots\}$$

is called the hyperbinary sequence and the ordered set

$$\left\{\frac{f(n)}{f(n+1)}\right\} = \left\{\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1}, \frac{1}{4}, \frac{4}{3}, \frac{3}{5}, \frac{5}{2}, \frac{2}{5}, \frac{5}{3}, \ldots\right\}$$

is called the Calkin–Wilf sequence where the ordering is on n ($n \geq 0$) and the first term in the Calkin–Wilf sequence is $\frac{f(0)}{f(1)}$.

Example 1. There are two hyperbinary expansions of 5, namely,
$5 = 4 + 1 = 2 + 2 + 1$. Thus $f(5) = 2$.
Similarly, there are three hyperbinary expansions of 4,
$4 = 4 = 2 + 2 = 2 + 1 + 1$. Thus $f(4) = 3$. Accordingly, $\frac{f(4)}{f(5)} = \frac{3}{2}$.

We note that $\{f(n)\}$ is sequence A002487 in [12].

3. Branches and diagonals of the Calkin–Wilf tree

Branches and diagonals are key features of the Calkin–Wilf tree and assist us in forming links with the Stern–Brocot tree.

Definition 4 (*Left And Right Branches*)**.** The left branch of a vertex, μ, called $\mathbb{L}_\mu$, is the set of vertices that is generated when an infinite number of right movements proceed from the left child of μ. $\mathbb{L}_\mu$ includes the left child of μ. Similarly, The right branch of a vertex, μ, called $\mathbb{R}_\mu$, is the set of vertices that is generated when an infinite number of left movements proceed from the right child of μ. $\mathbb{R}_\mu$ includes the right child of μ.

Throughout this paper $\mathbb{N}$ refers to the set of positive integers $\{1, 2, 3, \ldots\}$.

Example 2.

$$\mathbb{R}_{\frac{0}{1}} = \left\{\frac{1}{n}\middle| n \in \mathbb{N}\right\},$$
$$\mathbb{L}_{\frac{1}{0}} = \left\{\frac{n}{1}\middle| n \in \mathbb{N}\right\},$$
$$\mathbb{L}_{\frac{1}{1}} = \left\{\frac{2n-1}{2}\middle| n \in \mathbb{N}\right\}.$$

Definition 5 (*Left And Right Diagonals*)**.** For $j \in \mathbb{N}$, the jth left diagonal of the Calkin–Wilf tree, $\boldsymbol{L}_j$, is the set of all vertices found at the jth position in each level of the tree beginning at level $\lceil \log_2 j \rceil + 1$. Similarly, the jth right diagonal of the Calkin–Wilf tree, $\boldsymbol{R}_j$, is the set of all vertices found at the jth position from the rightmost end of each level of the tree beginning at level $\lceil \log_2 j \rceil + 1$.

$$\boldsymbol{L}_2 = \left\{\frac{n+1}{n}\middle| n \in \mathbb{N}\right\},$$
$$\boldsymbol{R}_1 = \left\{\frac{n}{1}\middle| n \in \mathbb{N}\right\} = \mathbb{L}_{\frac{1}{0}},$$
$$\boldsymbol{L}_1 = \left\{\frac{1}{n}\middle| n \in \mathbb{N}\right\} = \mathbb{R}_{\frac{0}{1}}.$$

Note that no diagonals, other than $\boldsymbol{R}_1$ and $\boldsymbol{L}_1$ are branches.

The following lemma is found in [5]. An inductive proof can be found in [1]. We offer a proof based on branching in the tree.

Lemma 1. *Let the concatenation of successive levels of the Calkin–Wilf tree, not including the* 0*th level, form a sequence. Then the denominator of each vertex in this sequence is the numerator of the next vertex in the sequence.*

Proof. Consider three kinds of vertices in the Calkin–Wilf tree:

- *Case* 1. The vertex is a left child.
  For this case, the vertex is succeeded by the right child of the same parent and by Definition 1, the denominator of the left child is the numerator of the right child.
- *Case* 2. The vertex is a right child but not the rightmost vertex of a level.
  Any such vertex $\frac{c}{d}$ must come from level 3 or lower in the Calkin–Wilf tree. Since, from Definition 4, every member of a left branch other than the first member is a right child, let $\frac{e}{f}$ be the vertex that commences the left branch containing $\frac{c}{d}$. It follows that $\frac{e}{f}$ is a left child.
  Let $\frac{a}{b}$ be the vertex whose left child is $\frac{e}{f}$. Then $e = a, f = a+b, c = a+k(a+b)$ and $d = a+b$, where k is the number of levels separating $\frac{c}{d}$ and $\frac{e}{f}$.
  Let also the vertex to the right of $\frac{c}{d}$ be $\frac{g}{h}$. Now $\frac{g}{h}$ is a left child with a branch commencing with $\frac{i}{j}$. By the symmetry of the tree structure, $\frac{i}{j}$ is a right child of $\frac{a}{b}$. Then $i = a+b, j = b, g = a+b$ and $j = b+k(a+b)$. Hence $d = g$ as required.
- *Case* 3. The vertex is the rightmost vertex of a level.
  As the rightmost vertices of each level form the left branch beginning with $\frac{1}{1}$ the result follows. □

The sequence described in Lemma 1 is of the form

$$\left\{\frac{g(n)}{g\,(n+1)}\right\}$$

for some function g and $n \geq 0$, where $g(0) = g\,(1) = 1$. Now the left and right children respectively of $\frac{g(n)}{g(n+1)}$ are $\frac{g(2n+1)}{g(2n+2)}$ and $\frac{g(2n+2)}{g(2n+3)}$. It follows from Definition 1 that for $n \geq 0$,

$$g\,(2n+1) = g(n) \quad \text{and}$$
$$g\,(2n+2) = g(n) + g\,(n+1)\,.$$

We now show that the sequence described in Lemma 1 is the hyperbinary sequence. Our proof is based on [5].

Theorem 2. *The Calkin–Wilf sequence is the concatenation of successive levels, starting with the first, of the Calkin–Wilf tree. That is,*

$$g(n) = f(n)$$

where from Definition 3, $f(n)$ is the number of ways of writing the integer n as a sum of powers of 2, each power being used at most twice; and $\frac{g(n)}{g(n+1)}$ is the sequence of Lemma 1.

Proof. We prove this by induction on n.
The theorem is true for $n = 0$.
Suppose the theorem is true for all integers $0, 1, \ldots, 2n$.
We show that

(1)

$$f\,(2n+1) = f(n). \tag{3.1}$$

Consider $f\,(2n+1)$. Any expansion of an odd number as a sum of powers of 2, (where each power is used at most twice) must contain a term with the value 1. There are no other terms

in the hyperbinary expansion with the value 1 (the only other possibility would be two terms each having the value 1, in which case the number would not be odd). Accordingly, within every hyperbinary expansion of $2n+1$ subtract 1. What remains is a subset of the hyperbinary expansions of $2n$. If we divide this subset of hyperbinary expansions of $2n$ by 2 we obtain the hyperbinary expansions for n. To show that these are all the hyperbinary expansions of n, consider the reverse process. Multiply all the hyperbinary expansions of n by 2. The result is a list of all the hyperbinary expansions of $2n$ that do not have 1 as a term. These are the only hyperbinary expansions of $2n$ that we are interested in because any that have two terms each with value 1 cannot, by Definition 3, have a further term with value 1 to form a hyperbinary expansion for $2n+1$. Adding 1 to each expansion in our list does not change the number of expansions and so $f(2n+1) = f(n)$.

(2)

$$f(2n+2) = f(n) + f(n+1). \tag{3.2}$$

Any hyperbinary expansion of $2n+2$ either contains two terms each with value 1 or no terms with value 1. Consider each of these cases.

(a) *Hyperbinary expansions of* $2n+2$ *that contain two terms each with value* 1.
Subtracting 2 from every expansion gives hyperbinary expansions of $2n$ that contain no terms with value 1. Further division by 2 gives hyperbinary expansions of n. To show that these are all the hyperbinary expansions of n, consider the reverse process. Multiply all the hyperbinary expansions of n by 2. The result is a list of all the hyperbinary expansions of $2n$ that do not have 1 as a term. Thus the number of hyperbinary expansions of $2n+2$ that contains two terms each with value 1 is equal to $f(n)$.

(b) *Hyperbinary expansions of* $2n+2$ *that contain no terms with value* 1.
Dividing all expansions by 2 gives a list of hyperbinary expansions of $n+1$. To show that this list represents all the hyperbinary expansions of $n+1$, consider the reverse process. Multiplying all hyperbinary expansions of $n+1$ gives hyperbinary expansions of $2n+2$ that contain no terms with value 1. Thus the number of hyperbinary expansions of $2n+2$ that contains no terms with value 1 is equal to $f(n+1)$.

Thus f and g possess identical recurrence formulae and initial values, establishing the theorem. □

Corollary 1. *For* $n > 0$, $f(2n) = f(2n-1) + f(2n+1)$.

Proof. From (3.1) and (3.2) for $n \geq 0$,

$$\begin{aligned} f(2n+2) &= f(2n+1) + f(n+1), \\ &= f(2n+1) + f(2n+3). \end{aligned} \tag{3.3}$$

The result follows by substituting $n-1$ for n in (3.3). □

Each entry in $\{f(n)\}$ is represented by the recursive formulae (3.1) and (3.2). The following corollary gives us an alternative set of recursive formulae.

Corollary 2.

$$f(n) = \begin{cases} f(n-1) - f(n-2), & \text{for } n \text{ odd and } n > 1 \\ f(n-1) + f\left(\dfrac{n}{2}\right), & \text{for } n \text{ even}. \end{cases}$$

Proof. For n odd, from Corollary 1, $f(2n+1) = f(2n) + f(2n-1)$. Substituting n for $2n+1$ gives the result.

For n even, from Corollary 1 and (3.1),

$$\begin{aligned} f(2n) &= f(2n-1) + f(2n+1), \\ &= f(2n-1) + f(n). \end{aligned} \tag{3.4}$$

Substituting n for $2n$ in (3.4) gives the result. □

We now show that $\{f(n)\}$ obeys a simple pattern:

Theorem 3. *For $n = 0, 1, 2, \ldots,$*

- *$f(3n)$ and $f(3n+1)$ are odd, and*
- *$f(3n+2)$ is even.*

Proof. We prove our result by induction on n.

Since $f(0) = f(1) = 1$ and $f(2) = 2$, the result is true for $n = 0$.

Suppose the result is true for all n up to and including $p-1$. Utilising (3.1) and (3.2), we consider two cases:

(i) p even $(=2k)$

As $k < p$,

- $f(3p) = f(2(3k-1)+2) = f(3k-1) + f(3k) = f(3(k-1)+2) + f(3k) = Odd.$
- $f(3p+1) = f(2 \cdot 3k + 1) = f(3k) = Odd.$
- $f(3p+2) = f(2 \cdot 3k + 2) = f(3k) + f(3k+1) = Even.$

(ii) p odd $(= 2k+1)$

As $k < p$,

- $f(3p) = f(2(3k+1)+1) = f(3k+1) = Odd.$
- $f(3p+1) = f(2(3k+1)+2) = f(3k+1) + f(3k+2) = Odd.$
- $f(3p+2) = f(2(3k+2)+1) = f(3k+2) = Even.$

It follows that our result is true for all n. □

Theorem 4. *Except for the first* 1*,*

(i) *Successive* 1*s in the hyperbinary sequence form the first and last terms of odd-sized palindromic sequences.*

(ii) *The kth such palindrome has middle term $f\left(3 \cdot 2^{k-1} - 1\right) = 2$.*

(iii) *The kth such palindrome has size $2^k + 1$ and commences with $f\left(2^k - 1\right) = 1$ and ends with $f\left(2^{k+1} - 1\right) = 1$.*

Proof. By Definition 1, beginning at level 1, 1s only appear as the numerator of the first vertex and the denominator of the last vertex in each level of the Calkin–Wilf tree. By Theorem 2, the Calkin–Wilf sequence is the concatenation of successive levels of the Calkin–Wilf tree. Consider the subsequence in this sequence that contains all of one level and the first term of the next level of the Calkin–Wilf tree. Since the kth level has 2^{k-1} vertices, by Theorem 1 and Definition 3, (i) and (iii) follow.

The middle term of the kth palindrome is

$$\begin{aligned} f\left(3 \cdot 2^{k-1} - 1\right) &= f\left(2\left(3 \cdot 2^{k-2} - 1\right) + 1\right), \\ &= f\left(3 \cdot 2^{k-2} - 1\right) \quad \text{by (3.1)}, \\ &\vdots \\ &= f\left(3 \cdot 2^{0} - 1\right), \\ &= f(2) = 2 \end{aligned}$$

establishing (ii). □

Corollary 3. *For $k > 0$ and $n = 0, 1, \ldots, 2^{k-1}$,*

$$f\left(2^{k-1} + n - 1\right) = f\left(2^k - n - 1\right).$$

Proof. Since $f\left(2^{k-1} - 1\right)$ and $f\left(2^k - 1\right)$ are the first and last terms respectively of a palindrome, the result follows. □

Theorem 5. (i) *For $k > 1$, $f\left(2^k + 1\right) = f\left(2^k - 2\right) = k$.*

(ii) $f\left(2^k\right) = k + 1$.

Proof. (i) From (3.1), $f(2^{k+1}+1) = f(2^k)$. From Corollary 3, for $n = 1$ and $k > 1$, $f(2^k) = f(2^{k+1}-2)$. The first equality follows. By Definition 1 and Theorem 2, $f(2^k - 1) = 1$ represents the numerator of the first vertex in level k of the tree. Thus for $k > 0$, $f(2^k - 2) = k$ represents the numerator of the last vertex in level $k - 1$ of the tree. The second equality follows and so the result follows.
(ii) From (3.1), $f(2^k) = f(2^{k+1} + 1)$. By (i), $f(2^{k+1} + 1) = k + 1$ and the result follows. □

4. Connectedness within the Calkin–Wilf tree

Our understanding of diagonals in the Calkin–Wilf tree is predicated on an understanding of connected vertices.

Definition 6 (*Connected Vertices*)**.** Two vertices found on different levels of the tree are connected if, beginning with the higher of the two vertices in the tree, we can move to the next lower level through its left or right child and continue in this way for each subsequent level until we arrive at the second vertex.

Definition 7 (*Paths*)**.** A path in the Calkin–Wilf tree is a series of zero or more downward left (L) and right (R) movements that connects a vertex with itself or another vertex. The null path connects a vertex to itself.

Not all vertices are connected by a path. However, when two vertices are connected by a path, that path is the only one that exists between the two vertices. For example, the path $RRL = R^2L$ connects the vertices $\frac{1}{2}$ and $\frac{5}{7}$. No path connects the vertices $\frac{3}{2}$ and $\frac{7}{3}$.

Definition 8 (*Connected Diagonals*)**.** Two diagonals are connected if for each $i > 0$, there is an identical path that connects the ith member of the first diagonal with the ith member of the second diagonal.

We now show that:

Theorem 6. *If the first vertex in a left (right) diagonal is connected by a path to the first vertex in another left (right) diagonal, then both diagonals are connected by the same path.*

Proof. We consider the case for left diagonals.

Let the first vertex in $\boldsymbol{L}_j$ be the jth vertex in level n. Let it be connected to the first vertex in $\boldsymbol{L}_k$ found in level $n + m$. Thus there are m movements in the path. Since at most one path can connect two vertices, the path could only have traversed through one of either the $2j$th or $(2j - 1)$th vertex in the $(n + 1)$th level according to whether we initially have a right or left movement. The next movement similarly has but two choices and so on for all movements until we reach the $(n + m)$th level.

Consider any other jth vertex in a level of the tree, that is, consider any other member of $\boldsymbol{L}_j$. If we were to trace the same path from any such member we would come, after the first movement, to the $2j$th or $(2j-1)$th vertex in the next level according to whether we initially had a right or left movement at the beginning of the path. Every other movement would also retrace the same positions within a level as when we first used the path. But this means that for each level at and below level n in the tree, the jth vertex is mapped to the kth vertex found m levels below. Since this mapping connects the first vertices in each left diagonal, it must also connect the second vertices in each left diagonal, and so on for corresponding ith vertices of both left diagonals. By Definition 8, the two left diagonals are connected.

The case for right diagonals is similar to that for left diagonals except that we refer to vertices in terms of their position from the *end* of the level rather than from the start of the level. □

Note that only left diagonals can be connected with left diagonals and right diagonals with right diagonals, and even then this is not true for every left or right diagonal. For example, $\boldsymbol{L}_3$ and $\boldsymbol{L}_7$ cannot be connected.

Corollary 4. *Every left diagonal is connected to* $\boldsymbol{L}_1$*; and every right diagonal is connected to* $\boldsymbol{R}_1$.

Proof. Since the vertex $\frac{1}{1}$ is connected to every other vertex and it is found on $\boldsymbol{L}_1$ and $\boldsymbol{R}_1$, by Theorem 6, the result follows. □

Theorem 7. (i) *The only left (right) diagonals connected to the jth left (right) diagonal, but lying below the jth left (right) diagonal, are the ones numbered:*

$$2^m j - \left(2^m - 1\right),\ 2^m j - \left(2^m - 2\right), \ldots, 2^m j$$

for all $m \in \mathbb{N}$.

(ii) *The only left (right) diagonals connected to the jth left (right) diagonal, but lying above the jth left (right) diagonal, are the ones numbered:*

$$\frac{j-k}{2^m} + 1$$

for suitable positive integers m and k where $1 \leq k \leq 2^m$.

(iii) *Any vertex connected to a vertex in the jth left (right) diagonal lies on one of the left (right) diagonals in* (i) *and* (ii).

Proof. We consider the case for left diagonals only. The case for right diagonals is similar to that for left diagonals except that we refer to vertices in terms of their position from the *end* of the level rather than from the start of the level.

As every vertex has two children and the first level consists of one vertex only, the kth level of the tree has 2^{k-1} vertices.

Consider left diagonals lying below $\boldsymbol{L}_j$.

- For $j > 1$:

  If the first vertex of $\boldsymbol{L}_j$ is on the kth level we must have:

  $$2^{k-2} + 1 \leq j \leq 2^{k-1}.$$

  (If $j \leq 2^{k-2}$ the first vertex of $\boldsymbol{L}_j$ would lie above the kth level).

  The children of the jth vertex on level l are the $(2j-1)$th and $(2j)$th vertices on level $l+1$ and so as $2^{k-1} < 2j - 1$, the $(2j-1)$th and $(2j)$th vertices on the $(k+1)$th level are the first vertices in $\boldsymbol{L}_{2j-1}$ and $\boldsymbol{L}_{2j}$ respectively.

  Hence $\boldsymbol{L}_j$ is connected to $\boldsymbol{L}_{2j-1}$ and $\boldsymbol{L}_{2j}$ by a single left or right downward step.

  Similarly, $\boldsymbol{L}_{2j-1}$ is connected to $\boldsymbol{L}_{4j-3}$ and $\boldsymbol{L}_{4j-2}$; and $\boldsymbol{L}_{2j}$ is connected to $\boldsymbol{L}_{4j-1}$ and $\boldsymbol{L}_{4j}$. These are also all the left diagonals that are connected to $\boldsymbol{L}_j$ by two downward steps. A continuation of this process proves (i). Also this process includes all possible paths that connect vertices in $\boldsymbol{L}_j$ to vertices lower down the tree, so as $j > 1$, (iii) holds for left diagonals below $\boldsymbol{L}_j$.
- For $j = 1$:

  The right child of $\frac{1}{1}$ is $\frac{2}{1}$, the first vertex of the second left diagonal. Thus $\boldsymbol{L}_1$ is connected to $\boldsymbol{L}_2$ by the same right downward step. The case for $j > 1$ showed that $\boldsymbol{L}_2$ is connected to all left diagonals below it and hence so is $\boldsymbol{L}_1$. Thus for $j = 1$, (i) and (iii) hold.

  We now consider left diagonals lying above $\boldsymbol{L}_j$, assuming they exist. Suppose $\boldsymbol{L}_i$ is connected to $\boldsymbol{L}_j$, where $i < j$. By part (i) of our theorem proven above, the only left diagonals connected to $\boldsymbol{L}_i$, but lying below $\boldsymbol{L}_i$, are the ones numbered:

  $$2^m i - \left(2^m - 1\right),\ 2^m i - \left(2^m - 2\right), \ldots, 2^m i$$

  for all $m \in \mathbb{N}$. Thus there must exist positive integers m and k where $k \leq 2^m$, for which $j = 2^m i - (2^m - k)$. That is, $i = \frac{j-k}{2^m} + 1$, where $1 \leq k \leq 2^m$. □

5. Identifying vertices within the Calkin–Wilf tree

Every positive integer j can be expressed as a finite alternating series of powers of 2 containing an odd number of summands. These powers of 2 are called the normalised additive factors of j and the series is called the normalised additive factorisation of j. The following definitions and results regarding normalised additive factorisation are based on Bates et al. [2] within which relevant proofs are found. They assist us in answering the question: *What is the* jth *vertex in the* nth *level of the Calkin–Wilf tree?*

Definition 9 (*Normalised Additive Factors*)**.** Let $b_0, b_1, \ldots, b_{2k}$ be the normalised additive factors of j defined through the following recursive algorithm:

- *Step* 1. If $j = 2^m$ then stop. The normalised additive factor of j is m. Otherwise, go to Step 2.
- *Step* 2. Set $j_0 = j$.
- *Step* 3. Let $b_i = \lceil \log_2 j_i \rceil$.
- *Step* 4. If $j_i = 2^r - 2^s$, then stop
  - For i odd, the normalised additive factors of j are $b_0, b_1, \ldots, b_i, s$.
  - For i even, the normalised additive factors of j are $b_0, b_1, \ldots, b_i, s+1, s$.

  Otherwise, go to Step 5.
- *Step* 5. Let $j_{i+1} = 2^{b_i} - j_i$.
- *Step* 6. Substitute $i + 1$ for i in Steps 3 to 5, and return to Step 3.

The following theorem guarantees that we can represent any integer, j, as a finite alternating power series based on powers of 2 consisting of an odd number of terms; and that consecutive powers in this power series are the consecutive normalised additive factors of j.

Theorem 8. *If $b_0, b_1, \ldots, b_s$ are the normalised additive factors of j, then $j = \sum_{i=0}^{s} (-1)^i 2^{b_i}$.*

Example 3. The number 6 has normalised additive factors $3, 2, 1$. Accordingly, it is normally additively factorised as $6 = 2^3 - 2^2 + 2^1$.

We now apply normalised additive factorisation to the Calkin–Wilf tree. The notation $[a_0, a_1, a_2, \ldots, a_n]$ is used to represent the continued fraction expansion

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}.$$

Theorem 9. *Let the normalised additive factors of j be $b_{j,0}, b_{j,1}, \ldots, b_{j,k}$. Then the jth vertex in the nth level of the Calkin–Wilf tree is given by the continued fraction:*

$$\left[b_{j,k}, \left(b_{j,k-1} - b_{j,k}\right), \left(b_{j,k-2} - b_{j,k-1}\right), \ldots, \left(b_{j,0} - b_{j,1}\right), \left(n - b_{j,0}\right)\right].$$

Proof. We prove this by induction on n.

The theorem is true in level 1 since $j = 1$ has only one normalised additive factor, namely, $b_{1,0} = 0$, and

$$\left[b_{1,0}, \left(n - b_{1,0}\right)\right] = [0, 1] = \frac{1}{1}$$

which is the first (and only) vertex in level 1.

Suppose for some level n the theorem is true. That is, for each j, where j is the position in a level, if $\left\{b_{j,0}, b_{j,1}, \ldots, b_{j,k}\right\}$ is the set of normalised additive factors of j, then the jth vertex in the nth level is given by:

$$\left[b_{j,k}, \left(b_{j,k-1} - b_{j,k}\right), \left(b_{j,k-2} - b_{j,k-1}\right), \ldots, \left(b_{j,0} - b_{j,1}\right), \left(n - b_{j,0}\right)\right]. \tag{5.1}$$

Now, by Corollary 4, there is a unique path that provides a one-to-one and onto correspondence between elements in $\boldsymbol{L}_1 = \langle [0, m] \rangle$ for $m \geq 1$ and elements in $\boldsymbol{L}_j$. By Theorem 6, if $[0, m]$ is connected by this path to the jth vertex in level n, then $[0, m+1]$ is connected by this same path to the jth vertex in level $n+1$.

Now the right child of $\frac{a}{b}$, represented by a downward right movement, is by Definition 1, $\frac{a+b}{b}$. Accordingly, if $\frac{a}{b} = [a_0, a_1, \ldots, a_k]$ then $1 + \frac{a}{b} = [a_0 + 1, a_1, \ldots, a_k]$. That is,

$$\frac{a+b}{b} = [a_0 + 1, a_1, \ldots, a_k]. \tag{5.2}$$

Similarly, the left child of $\frac{a}{b}$, represented by a left movement, is, by Definition 1, $\frac{a}{a+b}$.

Thus if $\frac{a}{b} = [a_0, a_1, \ldots, a_k]$, then $\frac{b}{a} = [0, a_0, a_1, \ldots, a_k]$ and $1 + \frac{b}{a} = \frac{a+b}{a} = [1, a_0, a_1, \ldots, a_k]$. Accordingly,

$$\frac{a}{a+b} = [0, 1, a_0, a_1, a_2, \ldots, a_k]. \tag{5.3}$$

Therefore the effect of a right or left movement is to leave the last term of the continued fraction of $\frac{a}{b}$ unchanged. Thus the path that connects the vertex $[0, m]$ to the jth vertex in level n, requires that m be the last term in the continued fraction of the jth vertex in level n as well. From (5.1), $m = n - b_{j,0}$. Now the path connecting $[0, m]$ to the jth vertex in level n will, by (5.2) and (5.3), have the following feature: each successive continued fraction along the path will differ from its predecessor either by

(i) the incrementing of the first term by 1, for a right movement, or,
(ii) inserting 0 and 1 as new first and second terms, while shifting all other terms two places to the right, for a left movement.

But the same path (that is the same set of left and right movements), is used to connect $[0, m + 1]$ to the jth vertex in level $n + 1$. Since $[0, m]$ and $[0, m + 1]$ only differ in their last terms, the jth vertex in level $n + 1$ must possess the same continued fraction as the jth vertex in level n, *except in its last term,* which is now $m + 1$. Since the jth vertex in level n is

$$\left[b_{j,k}, \left(b_{j,k-1} - b_{j,k}\right), \left(b_{j,k-2} - b_{j,k-1}\right), \ldots, \left(b_{j,0} - b_{j,1}\right), \left(n - b_{j,0}\right)\right]$$

it follows that the jth vertex in level $n + 1$ is

$$\left[b_{j,k}, \left(b_{j,k-1} - b_{j,k}\right), \left(b_{j,k-2} - b_{j,k-1}\right), \ldots, \left(b_{j,0} - b_{j,1}\right), \left(n + 1 - b_{j,0}\right)\right]$$

and so our theorem is proved. □

Example 4. What is the 15th vertex in the 5th level of the Calkin–Wilf tree?

Answer: The normalised additive factors of 15 are: 4, 1, 0. By Theorem 9, the 15th vertex in the 5th level of the Calkin–Wilf tree is $[0, 1, 3, 1] = [0, 1, 4] = \frac{4}{5}$.

Theorem 10. *The mth term in the Calkin–Wilf sequence is given by the continued fraction:*

$$\left[b_{j,k}, \left(b_{j,k-1} - b_{j,k}\right), \left(b_{j,k-2} - b_{j,k-1}\right), \ldots, \left(b_{j,0} - b_{j,1}\right), \left(\lceil \log_2 (m+1) \rceil - b_{j,0}\right)\right]$$

where $j = m + 1 - 2^{\lceil \log_2 (m+1) \rceil - 1}$, *and* $b_{j,0}, b_{j,1}, \ldots, b_{j,k}$ *are the normalised additive factors of* j.

Proof. By Theorem 2, the Calkin–Wilf sequence is the concatenation of successive levels of the Calkin–Wilf tree. Accordingly, the jth vertex in the nth level of the Calkin–Wilf tree is the $\left(2^{n-1} + j - 1\right)$th term in the Calkin–Wilf sequence.

Let $m = 2^{n-1} + j - 1$, then as $j \leq 2^{n-1}$, $n = \lceil \log_2 (m + 1) \rceil$. Then

$$j = m + 1 - 2^{\lceil \log_2 (m+1) \rceil - 1}.$$

Accordingly, by Theorem 9, with $n = \lceil \log_2 (m + 1) \rceil$ and $j = m + 1 - 2^{\lceil \log_2 (m+1) \rceil - 1}$, the mth term in the Calkin–Wilf sequence is given by the continued fraction:

$$\left[b_{j,k}, \left(b_{j,k-1} - b_{j,k}\right), \left(b_{j,k-2} - b_{j,k-1}\right), \ldots, \left(b_{j,0} - b_{j,1}\right), \left(\lceil \log_2 (m+1) \rceil - b_{j,0}\right)\right]. \quad \square$$

Corollary 5. *The mth term* $(m \geq 1)$ *in the Calkin–Wilf sequence is*

$$\frac{f(m-1)}{f(m)} = \left[b_{j,k}, \left(b_{j,k-1} - b_{j,k}\right), \left(b_{j,k-2} - b_{j,k-1}\right), \ldots, \left(b_{j,0} - b_{j,1}\right), \left(\lceil \log_2 (m+1) \rceil - b_{j,0}\right)\right]$$

where

(i) $f(m)$ *represents the number of ways of writing the integer* m *as a sum of powers of* 2, *each power being used at most twice and* $f(0) = 1$;

(ii) $j = m + 1 - 2^{\lceil \log_2(m+1) \rceil - 1}$; *and*
(iii) $b_{j,0}, b_{j,1}, \ldots, b_{j,k}$ *are the normalised additive factors of* j.

Proof. The result follows from combining Theorem 2 with Theorem 10. □

Remark 1. Corollary 5 gives us an alternative way of finding the number of ways of writing the integer m as a sum of powers of 2, each power being used at most twice. It is the denominator of the reduced rational number represented by the continued fraction

$$\left[b_{j,k}, \left(b_{j,k-1} - b_{j,k}\right), \left(b_{j,k-2} - b_{j,k-1}\right), \ldots, \left(b_{j,0} - b_{j,1}\right), \left(\lceil \log_2 (m+1) \rceil - b_{j,0}\right)\right]$$

where

(i) $j = m + 1 - 2^{\lceil \log_2(m+1) \rceil - 1}$; and
(ii) $b_{j,0}, b_{j,1}, \ldots, b_{j,k}$ are the normalised additive factors of j.

Example 5. The following question was posed by Stan Wagon as his "Problem of the Week" *in the* September–October 1997 edition of Quantum magazine and is mentioned in Calkin and Wilf [5]: "How many ways can 90 316 be written as $a + 2b + 4c + 8d + 16e + 32f + \cdots$, where the coefficients can be any of 0, 1, or 2?" In our notation, the question becomes, "What is f (90 316) ?" That is, how many ways can we express 90 316 as a sum of powers of 2, each power being used at most twice?

Answer: From Corollary 5, f (90 316) is the denominator of the 90 316th term in the Calkin–Wilf sequence. For $m = 90\,316$,

(i) $j = 90316 + 1 - 2^{\lceil \log_2(90316+1) \rceil - 1} = 24\,781$
(ii) 15, 13, 8, 6, 4, 2, 0 are the normalised additive factors of 24 781. Hence $k = 6$ and so $b_{24\,781,0} = 15$, $b_{24\,781,1} = 13$, $b_{24\,781,2} = 8$, $b_{24\,781,3} = 6$, $b_{24\,781,4} = 4$, $b_{24\,781,5} = 2$, $b_{24\,781,6} = 0$.
(iii) Accordingly,

$$\begin{aligned}&\left[b_{j,k}, \left(b_{j,k-1} - b_{j,k}\right), \left(b_{j,k-2} - b_{j,k-1}\right), \ldots, \left(b_{j,0} - b_{j,1}\right), \left(\lceil \log_2 (m+1) \rceil - b_{j,0}\right)\right] \\ &\quad = [0, 2, 2, 2, 2, 5, 2, 2], \\ &\quad = \frac{349}{843}.\end{aligned}$$

Thus the denominator of the 90316th term in the Calkin–Wilf sequence is 843. That is, f (90 316), the number of ways of writing the integer 90 316 as a sum of powers of 2, each power being used at most twice, is 843.

6. The Stern–Brocot tree

We recall the definition of the Stern–Brocot tree and its diagonals from [2].

Definition 10 (*Mediant*)**.** A mediant is a fraction that is formed from two other fractions by adding their numerators to obtain a new numerator and adding their denominators to obtain a new denominator.

Definition 11 (*Stern–Brocot Tree*)**.** The Stern–Brocot tree is a binary tree of fractions (called terms) with the pseudofractions $\frac{0}{1}$ and $\frac{1}{0}$ constituting level 0. A new level is formed when all terms in preceding levels are arranged in ascending order and mediants are obtained from consecutive terms of this sequence. These mediants, when arranged in ascending order, constitute terms of the new level.

Fig. 2 shows levels 0–5 of the Stern–Brocot tree.

Definition 12 (*Left diagonals*)**.** We define $\frac{0}{1}$ and $\frac{1}{0}$ as the zeroth and first terms respectively from the left in the zeroth level.

(i) For $k \in \mathbb{N}$, the kth **left diagonal** of the tree, $\mathcal{L}_k$, is the sequence made up of each kth term from each level beginning at the first level.
(ii) The **first**$^+$ **left diagonal** of the tree, $\mathcal{L}_{1^+}$, is the first left diagonal with the term $\frac{1}{0}$.

Left diagonals are read from upper right to lower left of the tree.

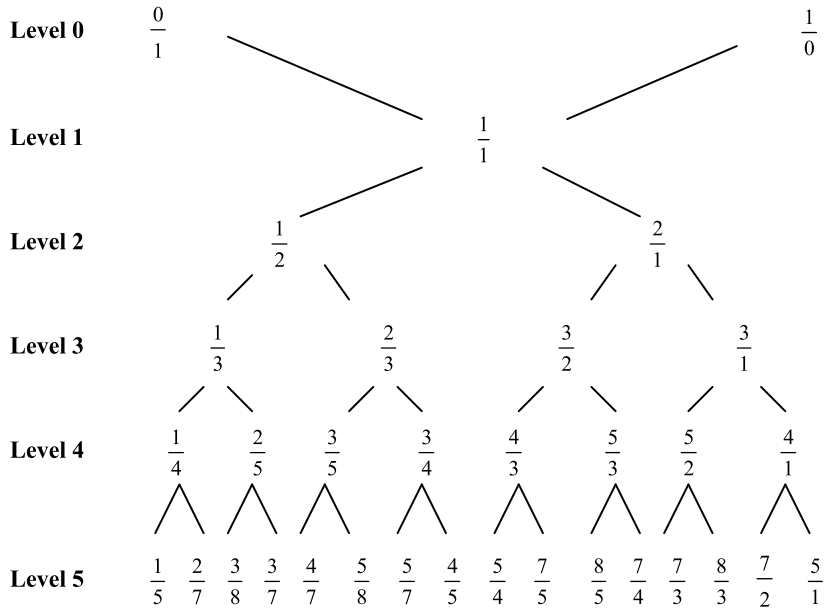


Fig. 2. Levels 0–5 of the Stern–Brocot tree.

Definition 13 (*Right Diagonals*)**.** We define $\frac{1}{0}$ and $\frac{0}{1}$ as the zeroth and first terms respectively from the right in the zeroth level.

(i) For $k \in \mathbb{N}$, the kth **right diagonal** of the tree, $\mathcal{R}_k$, is the sequence made up of each kth term taken from the end of each level beginning at the first level.
(ii) The **first**$^+$ **right diagonal** of the tree, $\mathcal{R}_{1^+}$, is the first right diagonal with the term $\frac{0}{1}$.

Right diagonals are read from upper left to lower right of the tree.

7. Mapping vertices from the Calkin–Wilf tree to the Stern–Brocot tree

In this section we show, by normalised additive factorisation, a convenient way to convert the jth vertex in the nth level of the Calkin–Wilf tree into the jth entry in the nth level of the Stern–Brocot tree. We also show how to locate a particular vertex within the Calkin–Wilf tree.

Theorem 11. *For any vertex* $\frac{c}{d} = [a_0, a_1, \ldots, a_m]$ *where* $\frac{c}{d}$ *is found in the nth level of the Calkin–Wilf tree,*

$$n = a_0 + a_1 + \cdots + a_m. \tag{7.1}$$

Proof. Whenever we perform a right or left movement from a vertex to move to the next level we utilise either (5.2) or (5.3). Since [0, 1] is found in level 1, for which (7.1) holds, every movement to the next lower level is associated with an incremental increase to (7.1). □

We note that since there is an odd number of normalised additive factors associated with every integer, then k is even in Theorem 9. Therefore,

$$\left[b_{j,k}, \left(b_{j,k-1} - b_{j,k}\right), \left(b_{j,k-2} - b_{j,k-1}\right), \ldots, \left(b_{j,0} - b_{j,1}\right), \left(n - b_{j,0}\right)\right] \tag{7.2}$$

has an even number of terms. The following remarks can now be made:

Remark 2. Every positive rational number can be represented by a long and short form continued fraction expansion. Let δ be a positive rational number such that $\delta = [a_0, a_1, \ldots, a_m]$ where $a_0 \geq 0$, $a_i > 0$ for $i = 1, \ldots, m-1$, and $a_m > 1$ for $m > 0$. Then $[a_0, a_1, \ldots, a_m]$ is the short form and

$[a_0, a_1, \ldots, a_m - 1, 1]$ is the long form of δ. Accordingly,

$$\left[b_{j,k}, \left(b_{j,k-1} - b_{j,k}\right), \left(b_{j,k-2} - b_{j,k-1}\right), \ldots, \left(b_{j,0} - b_{j,1}\right), \left(n - b_{j,0}\right)\right]$$

is either the long or short form of the continued fraction of the jth vertex in the nth level of the Calkin–Wilf tree according to whether the long or short form possesses an even number of terms in its continued fraction.

Remark 3. For $n > 0$, the jth vertex in the nth level of the Calkin–Wilf tree is given by the continued fraction:

$$\left[b_{j,k}, \left(b_{j,k-1} - b_{j,k}\right), \left(b_{j,k-2} - b_{j,k-1}\right), \ldots, \left(b_{j,0} - b_{j,1}\right), \left(n - b_{j,0}\right)\right]$$

and, the jth entry in the nth level of the Stern–Brocot tree is given by the continued fraction:

$$\left[0, n - b_{j,0} - 1, \left(b_{j,0} - b_{j,1}\right), \left(b_{j,1} - b_{j,2}\right), \ldots, \left(b_{j,k-1} - b_{j,k}\right), b_{j,k} + 1\right]$$

where $b_{j,0}, b_{j,1}, \ldots, b_{j,k}$ are the normalised additive factors of j.

Thus to convert the jth vertex in the nth level of the Calkin–Wilf tree into the jth entry in the nth level of the Stern–Brocot tree, do the following:

- *Step* 1. Express the vertex as a continued fraction possessing an even number of terms. (This will be either the long or short form continued fraction of the vertex), then
- *Step* 2. Add 1 to the first term and subtract 1 from the last term, then
- *Step* 3. Rewrite backwards its continued fraction, then
- *Step* 4. Add a first term 0.

We note that Remark 3 provides another demonstration that all rationals appear in the Calkin–Wilf tree, as every vertex in the Calkin–Wilf tree is a modification of every entry from the Stern–Brocot tree.

Example 6. The 7th vertex in the 4th level of the Calkin–Wilf tree is: $\frac{3}{4} = [0, 1, 3]$. What is the 7th entry in the 4th level of the Stern–Brocot tree?

Answer: From Remark 3,
Step 1. $\frac{3}{4} = [0, 1, 2, 1]$
Step 2. $[1, 1, 2, 0]$
Step 3. $[0, 2, 1, 1]$
Step 4. $[0, 0, 2, 1, 1] = [2, 2] = \frac{5}{2}$.

Theorem 12. *Let $\alpha = [a_0, a_1, \ldots, a_m]$, where m is odd, be a vertex in the Calkin–Wilf tree and j be the position in level $\sum_{i=0}^{m} a_i$ that α occupies. If*

$$j = \sum_{i=0}^{k} (-1)^i 2^{b_{j,i}}$$

is the normalised additive factorisation of j, then

$$b_{j,i} = \sum_{l=0}^{k-i} a_l$$

and $m = k + 1$.

Proof. From Remark 2,

$$[a_0, a_1, \ldots, a_m] = \left[b_{j,k}, \left(b_{j,k-1} - b_{j,k}\right), \left(b_{j,k-2} - b_{j,k-1}\right), \ldots, \left(b_{j,0} - b_{j,1}\right), \left(n - b_{j,0}\right)\right] \quad (7.3)$$

where both sides in (7.3) contain an even number of terms. But the right-hand side of (7.3) possesses $k + 2$ terms with all terms positive except the first term which can also be equal to zero. It follows that $m = k + 1$.

Equating terms on the left- and right-hand sides of (7.3) yields:

$$\begin{aligned}
b_{j,k} &= a_0,\\
b_{j,k-1} &= a_0 + a_1,\\
b_{j,k-2} &= a_0 + a_1 + a_2,\\
&\vdots\\
b_{j,1} &= a_0 + a_1 + \cdots + a_{k-1},\\
b_{j,0} &= a_0 + a_1 + \cdots + a_k.
\end{aligned}$$

That is,

$$b_{j,i} = \sum_{l=0}^{k-i} a_l = \sum_{l=0}^{m-1-i} a_l. \quad \square$$

Example 7. Where is the vertex $\frac{3}{7}$ located in the Calkin–Wilf tree?
Answer: From Theorem 11, $\frac{3}{7} = [0, 2, 2, 1]$ is found on level $0 + 2 + 2 + 1 = 5$.
By Theorem 12,

$$b_{j,0} = \sum_{l=0}^{2} a_l = 4,$$

$$b_{j,1} = \sum_{l=0}^{1} a_l = 2,$$

$$b_{j,2} = \sum_{l=0}^{0} a_l = 0.$$

Therefore within level 5, the vertex $\frac{3}{7}$ is found at position

$$j = 2^4 - 2^2 + 2^0 = 13.$$

8. Mapping terms from the Stern–Brocot tree to the Calkin–Wilf tree

In this section we perform the reverse map to that found in Remark 3 of the previous section. That is, we convert the jth entry in the nth level of the Stern–Brocot tree into the jth vertex in the nth level of the Calkin–Wilf tree. We represent the steps through the following Remark:

Remark 4. To convert the jth entry in the nth level of the Stern–Brocot tree into the jth vertex in the nth level of the Calkin–Wilf tree:

- *Step* 1. Take its reciprocal.
- *Step* 2. Express this reciprocal as a continued fraction possessing an even number of terms. (This will be either the long or short form), then
- *Step* 3. Rewrite backwards this continued fraction, then
- *Step* 4. Subtract 1 from the first term and add 1 to the last term.

We now use these steps to reverse Example 6.

Example 8. The 7th entry in the 4th level of the Stern–Brocot tree is: $\frac{5}{2} = [2, 2]$. What is the 7th vertex in the 4th level of the Calkin–Wilf tree?
Answer: From Remark 4,
Step 1. $\frac{2}{5} = [0, 2, 2]$
Step 2. $[0, 2, 1, 1]$
Step 3. $[1, 1, 2, 0]$
Step 4. $[0, 1, 2, 1] = \frac{3}{4}$.

9. Mapping diagonals and branches between the Calkin–Wilf and Stern–Brocot trees

Using normalised additive factorisation we can now formulate all diagonals in the Calkin–Wilf tree.

Lemma 2. *If* $[a_0, a_1, \ldots, a_m]$ *is the first vertex in a diagonal of the Calkin–Wilf tree, then the remaining vertices of the diagonal are successively* $[a_0, a_1, \ldots, a_m - 1, t]$ *for* $t = 2, 3, \ldots$.

Proof. We consider left and right diagonals

(1) *Left diagonals.*
Each vertex must appear on one and only one left diagonal. Let μ be the jth vertex in the level n of the Calkin–Wilf tree. By Theorem 9,

$$\mu = \left[b_{j,k}, \left(b_{j,k-1} - b_{j,k}\right), \left(b_{j,k-2} - b_{j,k-1}\right), \ldots, \left(b_{j,0} - b_{j,1}\right), \left(n - b_{j,0}\right)\right] \tag{9.1}$$

where
- $b_{j,0}, b_{j,1}, \ldots, b_{j,k}$ are the normalised additive factors of j;
- all terms in the right-hand side of (9.1) after the first term are positive integers; and
- $b_{j,k} \geq 0$.

Now (9.1) is either the long or short form continued fraction of μ. There are two cases to consider:

(a) *The right-hand side of* (9.1) *is the short form of* μ, *that is,* $n - b_{j,0} > 1$.
Thus the jth vertex in level $n - 1$ is
$$\left[b_{j,k}, \left(b_{j,k-1} - b_{j,k}\right), \left(b_{j,k-2} - b_{j,k-1}\right), \ldots, \left(b_{j,0} - b_{j,1}\right), \left(n - 1 - b_{j,0}\right)\right]$$
where $n - 1 - b_{j,0} \geq 1$. But this means that μ is no higher than the second vertex in $\boldsymbol{L}_j$. Thus μ does not commence $\boldsymbol{L}_j$.

(b) *The right-hand side of* (9.1) *is the long form of* μ, *that is,* $n - b_{j,0} = 1$.
Thus $n - 1 - b_{j,0} = 0$ and the jth vertex in level $n - 1$ does not exist. Hence μ must be the first vertex in $\boldsymbol{L}_j$ with successive vertices
$$\left[b_{j,k}, \left(b_{j,k-1} - b_{j,k}\right), \left(b_{j,k-2} - b_{j,k-1}\right), \ldots, \left(b_{j,0} - b_{j,1}\right), \left(n - b_{j,0} + t\right)\right]$$
for $t = 1, 2, \ldots$. The result follows.

(2) *Right diagonals.*
Each vertex must appear on one and only one right diagonal.
Let μ be the jth vertex from the end of level n of the Calkin–Wilf tree. From Theorem 1,

$$\mu = \left[0, b_{j,k}, \left(b_{j,k-1} - b_{j,k}\right), \left(b_{j,k-2} - b_{j,k-1}\right), \ldots, \left(b_{j,0} - b_{j,1}\right), \left(n - b_{j,0}\right)\right] \tag{9.2}$$

where
- $b_{j,0}, b_{j,1}, \ldots, b_{j,k}$ are the normalised additive factors of j;
- all terms in the right-hand side of (9.2) after the second term are positive integers; and
- $b_{j,k} \geq 0$.

The proof now follows that given in (1) (a) and (b) except that we make reference to $\boldsymbol{R}_j$ and not $\boldsymbol{L}_j$.

The result follows. □

Lemma 3. *Let* $[a_0, a_1, \ldots, a_k]$ *be the short form of a vertex,* μ, *in the Calkin–Wilf tree.*

(i) *For* k *odd,* μ *is found in the left half of the tree.*
(ii) *For* k *even,* μ *is found in the right half of the tree.*

Proof. Let $\mu = \frac{p}{q} = [a_0, a_1, \ldots, a_k]$ where $[a_0, a_1, \ldots, a_k]$ is the short form continued fraction expansion of μ. From Definition 1,

(1) The left child of μ has short form continued fraction expansion

$$\begin{aligned}[d_0, d_1, \ldots, d_m] = \frac{p}{p+q} &= \frac{1}{1 + \frac{q}{p}} = \frac{1}{1 + \frac{1}{\frac{p}{q}}}, \\ &= [0, 1, a_0, a_1, \ldots, a_k], \\ &= \begin{cases} [0, 1, a_0, a_1, \ldots, a_k] \mid a_0 > 0 \\ [0, 1 + a_1, a_2 \ldots, a_k] \mid a_0 = 0. \end{cases}\end{aligned}$$

That is, for k odd (even), m is odd (even).

(2) The right child of μ has short form continued fraction expansion

$$\begin{aligned}[d_0, d_1, \ldots, d_m] &= \frac{p+q}{q} = 1 + \frac{p}{q}, \\ &= [a_0 + 1, a_1, \ldots, a_k]\,.\end{aligned}$$

That is, for k odd (even), $m\ (= k)$ is odd (even).

Combining (1) and (2), since all vertices in the left half of the tree are descended from $\frac{1}{2}$, which has short form continued fraction expansion $[0, 2]$, the result in (i) follows. Similarly, since all vertices in the right half of the tree are descended from 2, which has short form continued fraction expansion $[2]$, the result in (ii) follows. □

Theorem 13. *Let* $M = \{[a_0, a_1, \ldots, a_k, t] \mid t \geq 1\}$ *represent a diagonal in the Calkin–Wilf tree, commencing with* $[a_0, a_1, \ldots, a_k + 1]$. *Then*

(i) *For* k *even,* M *is a left diagonal.*
(ii) *For* k *odd,* M *is a right diagonal.*

Proof. (i) Left diagonals have their first vertices in the right half of the tree and all other vertices in the left half of the tree. Combining Lemmas 2 and 3 the short form of every vertex in a left diagonal other than the first vertex is of the form $[a_0, a_1, \ldots, a_k, t]$, $t = 2, 3, \ldots$ where k is even. The first vertex has short form $[a_0, a_1, \ldots, a_k + 1]$ which, for k even, is in the right half of the tree by Lemma 3.

(ii) Right diagonals have their first vertices in the left half of the tree and all other vertices in the right half of the tree. Combining Lemmas 2 and 3 again, the short form of every vertex in a right diagonal other than the first vertex is of the form $[a_0, a_1, \ldots, a_k, t]$, $t = 2, 3, \ldots$, where k is odd. The first vertex has short form $[a_0, a_1, \ldots, a_k + 1]$ which, for k odd, is in the left half of the tree by Lemma 3. □

Definition 14 (*Left and Right Branches of the Stern–Brocot Tree*)**.** In the Stern–Brocot tree, the set of all mediants possessing a common parent μ and whose elements are smaller than μ is called the *left branch* of μ; the set of all mediants possessing a common parent μ and whose elements are greater than μ is called the *right branch* of μ.

The following theorems and their proofs can be found in [3].

Theorem 14. *Let* $[a_0, a_1, a_2, \ldots, a_k]$ *be the short form of* μ. *Then, in the Stern–Brocot tree,*

(i) *for* k *odd,*
the right branch of μ *is the set* $\{[a_0, a_1, a_2, \ldots, a_k - 1, 1, t] \mid t \geq 1\}$ *and*
the left branch of μ *is the set* $\{[a_0, a_1, a_2, \ldots, a_k, t] \mid t \geq 1\}$;
(ii) *for* k *even,*
the right branch of μ *is the set* $\{[a_0, a_1, a_2, \ldots, a_k, t] \mid t \geq 1\}$ *and*
the left branch of μ *is the set* $\{[a_0, a_1, a_2, \ldots, a_k - 1, 1, t] \mid t \geq 1\}$.

Theorem 15. $[a_0, a_1, a_2, \ldots, a_n]$ *is found on level* $\sum_{i=0}^{n} a_i$ *of the Stern–Brocot tree.*

Corollary 6. *Left (Right) diagonals of the Calkin–Wilf tree are right (left) branches of the Stern–Brocot tree.*

Proof. A comparison of Theorems 13 and 14 yields the result. □

Theorem 16. *Left (Right) branches of the Calkin–Wilf tree are right (left) diagonals of the Stern–Brocot tree.*

Proof. By Definition 1, for $a > b$,

$$\mathbb{R}_{\frac{a-b}{b}} = \left\langle \frac{a}{b}, \frac{a}{a+b}, \frac{a}{2a+b}, \ldots \right\rangle = \mathcal{L}_k$$

where $\mathcal{L}_k$ is the left diagonal that commences with $\frac{a}{b}$ in the Stern–Brocot tree. Similarly, for $b > a$,

$$\mathbb{L}_{\frac{a}{b-a}} = \left\langle \frac{a}{b}, \frac{a+b}{b}, \frac{a+2b}{b}, \ldots \right\rangle = \mathcal{R}_k$$

where $\mathcal{R}_k$ is the right diagonal that commences with $\frac{a}{b}$ in the Stern–Brocot tree. □

10. Successors and predecessors in the Calkin–Wilf sequence

The following is a proof of a result attributed to Moshe Newman and mentioned without proof in [10].

Theorem 17. *Let x be a term in the Calkin–Wilf sequence and $\{x\}$ its fractional part. Then*

$$x_s = \frac{1}{\lfloor x \rfloor + 1 - \{x\}}$$

where x_s represents the term that succeeds x in the sequence.

Proof. There are two cases to consider:

(1) $x \notin \mathbb{N}$. That is, excluding level 0, x is not the last term in a level of the Calkin–Wilf tree.
Let $\frac{a}{b}$ be a root for x and x_s. That is, x and x_s are the kth terms respectively in the left and right branches of $\frac{a}{b}$. It follows that x is the kth term in the left branch of $\frac{a}{b}$ that commences with $\frac{a}{a+b}$. Hence

$$x = \frac{m}{n} = \frac{k(a+b)+a}{a+b}.$$

Thus $n = a+b$, $k = \left\lfloor \frac{m}{n} \right\rfloor$, $a = n\left\{\frac{m}{n}\right\}$ and $b = n - n\left\{\frac{m}{n}\right\}$.
Similarly, x_s is the kth term in the right branch of $\frac{a}{b}$ that commences with $\frac{a+b}{b}$. Hence

$$\begin{aligned} x_s &= \frac{a+b}{k(a+b)+b}, \\ &= \frac{n}{\left\lfloor \frac{m}{n} \right\rfloor n + n - n\left\{\frac{m}{n}\right\}}, \\ &= \frac{1}{\lfloor x \rfloor + 1 - \{x\}}. \end{aligned}$$

(2) $x \in \mathbb{N}$. That is, excluding level 0, x is the last term in a level of the Calkin–Wilf tree.
Let $x = \frac{i}{1}$, $i = 1, 2, \ldots$. Then

$$x_s = \frac{1}{i+1} = \frac{1}{\lfloor i \rfloor + 1 - \{i\}} = \frac{1}{\lfloor x \rfloor + 1 - \{x\}}.$$ □

Theorem 18. *Let x be a term in the Calkin–Wilf sequence, after the first. Then*

$$x_p = \begin{cases} \dfrac{\frac{1}{x} - 1}{1}, & \text{for } \dfrac{1}{x} \in \mathbb{N} \\ \left\lfloor \dfrac{1}{x} \right\rfloor + 1 - \left\{ \dfrac{1}{x} \right\}, & \text{otherwise} \end{cases}$$

where x_p represents the term that precedes x in the sequence.

Proof. Replacing x by x_p in Theorem 17 and inverting gives $\frac{1}{x} = \lfloor x_p \rfloor + 1 - \{x_p\}$.
There are two cases to consider:

(1) $\frac{1}{x} \in \mathbb{N}$. Then $\{x_p\} = 0$ and $\lfloor x_p \rfloor = \frac{1}{x} - 1$, so $x_p = \frac{\frac{1}{x}-1}{1}$.
(2) $\frac{1}{x} \notin \mathbb{N}$. Then $\left\lfloor \frac{1}{x} \right\rfloor = \lfloor x_p \rfloor$ and $\left\{\frac{1}{x}\right\} = 1 - \{x_p\}$, so $x_p = \left\lfloor \frac{1}{x} \right\rfloor + 1 - \left\{\frac{1}{x}\right\}$. □

Theorem 19. *Let x be a term in the Calkin–Wilf sequence, after the first. Then*

$$\left(\frac{1}{x}\right)_s = \begin{cases} \dfrac{1}{x_p+2} & \text{for } \dfrac{1}{x} \in \mathbb{N} \\ \dfrac{1}{x_p} & \text{otherwise} \end{cases} \quad \text{and}$$

$$\left(\frac{1}{x}\right)_p = \begin{cases} \dfrac{1}{x_s} - 2 & \text{for } x \in \mathbb{N} \\ \dfrac{1}{x_s} & \text{otherwise.} \end{cases}$$

Proof. The result follows from Theorems 17 and 18. □

11. Successors and predecessors in the Stern–Brocot sequence

The Stern–Brocot sequence is defined as:

Definition 15 (*Stern–Brocot Sequence*)**.** Let $A_0 = \langle a_{0,1}, a_{0,2}\rangle$ represent a sequence consisting of two arbitrary integers, and define for $n \geq 1$,

$$A_n = \langle a_{n,1}, a_{n,2}, \ldots, a_{n,2^n+1}\rangle$$

as the sequence for which, for $k \geq 1$,

$$a_{n,2k-1} = a_{n-1,k} \quad \text{and}$$
$$a_{n,2k} = a_{n-1,k} + a_{n-1,k+1}.$$

Similarly let $B_0 = \langle b_{0,1}, b_{0,2}\rangle$ represent another sequence consisting of two arbitrary terms, and define for $n \geq 1$,

$$B_n = \langle b_{n,1}, b_{n,2}, \ldots, b_{n,2^n+1}\rangle$$

as another sequence for which,

$$b_{n,2k-1} = b_{n-1,k} \quad \text{and}$$
$$b_{n,2k} = b_{n-1,k} + b_{n-1,k+1}.$$

Then the sequence defined by $H_n = \langle h_{n,1}, h_{n,2}, \ldots, h_{n,2^n+1}\rangle$, where $h_{n,i} = \frac{a_{n,i}}{b_{n,i}}$ and $a_{0,1} = b_{0,2} = 0$ and $a_{0,2} = b_{0,1} = 1$, is called the *Stern–Brocot Sequence* of order n. It represents the ordered sequence containing both the first n generations of mediants based on H_0, and the terms of H_0 itself.

We want to develop a set of results for predecessors and successors in the Stern–Brocot and Calkin–Wilf sequences based on continued fractions. But first, we extend our definition of continued fractions to include terms with value 0.

Definition 16 (*Continued Fractions Containing The Term 0*)**.**

(i) For $a_0 \geq 0$, $[a_0, 0] = \frac{1}{0}$.
(ii) For $n > 0$, $[a_0, a_1, a_2, \ldots, a_n, 0] = [a_0, a_1, a_2, \ldots, a_{n-1}]$
(iii) For $n > k+1$, $[a_0, a_1, a_2, \ldots, a_k, 0, a_{k+2}, \ldots, a_n] = [a_0, a_1, a_2, \ldots, a_{k-1}, a_k + a_{k+2}, a_{k+3}, \ldots, a_n]$.

Note that Definition 16 (ii) amounts to using $\frac{1}{a_i + \frac{1}{0}} = 0$ with the standard definition.

Lemma 4. *If $x = [0, a_1, a_2, \ldots]$ and $a_1 > 0$, then $1 - x = [0, 1, a_1 - 1, a_2, a_3, \ldots]$.*

Proof. Let $\beta = [0, a_2, a_3, \ldots]$, then $x = [0, a_1, a_2, \ldots] = \frac{1}{a_1+\beta}$. Accordingly,

$$1 - x = \frac{a_1 + \beta - 1}{a_1 + \beta} = \frac{1}{1 + \frac{1}{(a_1-1)+\beta}} = [0, 1, a_1 - 1, a_2, a_3, \ldots]. \quad \square$$

Theorem 20. *Let $[a_0, a_1, a_2, \ldots, a_n]$ be the short form of x. Then*

$$x_s = \begin{cases} [0, a_0 + 1], & \text{for } n = 0 \\ [0, a_0, 1, a_1 - 1, a_2, \ldots, a_n], & \text{for } n > 0 \end{cases}$$

where x_s represents the term that succeeds x in the Calkin–Wilf sequence.

Proof. (i) For $n = 0$, by Theorem 17.
(ii) For $n > 0$, from Theorem 17 and Lemma 4,

$$\begin{aligned} x_s &= \frac{1}{\lfloor x \rfloor + 1 - \{x\}}, \\ &= \frac{1}{a_0 + [0, 1, a_1 - 1, a_2, \ldots, a_n]}, \\ &= \frac{1}{[a_0, 1, a_1 - 1, a_2, \ldots, a_n]}, \\ &= [0, a_0, 1, a_1 - 1, a_2, \ldots, a_n]. \quad \square \end{aligned}$$

Theorem 21. *Let $[a_0, a_1, a_2, \ldots, a_n]$ be the short form of x. Then*

$$x_p = \begin{cases} \dfrac{a_1 - 1}{1} & \text{for } a_0 = 0,\ n = 1 \\ [a_1, 1, a_2 - 1, a_3, \ldots, a_n] & \text{for } a_0 = 0,\ n > 1 \\ [0, 1, a_0 - 1, a_1, \ldots, a_n] & \text{for } a_0 > 0 \end{cases}$$

where x_p represents the term that precedes x in the Calkin–Wilf sequence.

Proof. (i) For $a_0 = 0, n = 1$, the result follows from the case $\frac{1}{x} \in \mathbb{N}$ in Theorem 18.
(ii) For $a_0 = 0, n > 1$, we have $x = [0, a_1, \ldots, a_n]$. Using Theorem 18 and Lemma 4,

$$\begin{aligned} x_p &= \left\lfloor \frac{1}{x} \right\rfloor + 1 - \left\{ \frac{1}{x} \right\}, \\ &= a_1 + 1 - [0, a_2, \ldots, a_n], \\ &= [a_1, 1, a_2 - 1, a_3, \ldots, a_n]. \end{aligned}$$

(iii) For $a_0 > 0$, using Theorem 18 and Lemma 4 again,

$$\begin{aligned} x_p &= \left\lfloor \frac{1}{x} \right\rfloor + 1 - \left\{ \frac{1}{x} \right\}, \\ &= 0 + 1 - [0, a_0, \ldots, a_n], \\ &= [0, 1, a_0 - 1, a_1, \ldots, a_n]. \quad \square \end{aligned}$$

We are now able to explore successors and predecessors in the Stern–Brocot sequence.

Theorem 22. *Let x_p and x_s respectively represent terms that precede and succeed x in the Stern–Brocot sequence of length $2^m + 1$. If $[a_0, a_1, a_2, \ldots, a_n]$ is the short form of x, $t = 0, 1, 2, \ldots$ and $m = t + \sum_{i=0}^{n} a_i$, then, where x_p and x_s exist,*

(i) *For n odd,*

$$x_p = [a_0, a_1, a_2, \ldots, a_n, t] \quad \textit{and}$$
$$x_s = [a_0, a_1, a_2, \ldots, a_{n-1}, a_n - 1, 1, t]\,.$$

(ii) *For n even,*

$$x_p = [a_0, a_1, a_2, \ldots, a_{n-1}, a_n - 1, 1, t] \quad \textit{and}$$
$$x_s = [a_0, a_1, a_2, \ldots, a_n, t]\,.$$

Proof. There are two cases to consider:

(1) $t > 0$:

From Theorems 14 and 15, $[a_0, a_1, a_2, \ldots, a_n, t]$ and $[a_0, a_1, a_2, \ldots, a_n - 1, 1, t]$ are adjacent terms in level $t + \sum_{i=0}^{n} a_i$ of the Stern–Brocot tree. Hence $[a_0, a_1, a_2, \ldots, a_n, t]$ and $[a_0, a_1, a_2, \ldots, a_n - 1, 1, t]$ are terms found either side of $[a_0, a_1, a_2, \ldots, a_n]$ in the Stern–Brocot sequence of length $2^m + 1$ where $m = t + \sum_{i=0}^{n} a_i$. By Theorem 14:

(a) For n odd, $[a_0, a_1, a_2, \ldots, a_n, t]$, $[a_0, a_1, a_2, \ldots, a_n]$ and $[a_0, a_1, a_2, \ldots, a_n - 1, 1, t]$ are successive terms in the Stern–Brocot sequence of length $2^m + 1$. For the particular case $x = \frac{1}{0} = [0, 0]$ we have $x_p = [t]$ in accordance with our result, while x_s does not exist.

(b) For n even the order of successive terms found in (i) is reversed; and for the particular case $x = \frac{0}{1} = [0]$ we have $x_s = [0, t]$ in accordance with our result, while x_p does not exist.

(2) $t = 0$:

We need to find the terms found either side of $[a_0, a_1, a_2, \ldots, a_n]$ in the Stern–Brocot sequence of length $2^m + 1$ where $m = \sum_{i=0}^{n} a_i$. By Theorem 15, $[a_0, a_1, a_2, \ldots, a_n]$ is found in level $\sum_{i=0}^{n} a_i$, so these terms are the parents of $[a_0, a_1, a_2, \ldots, a_n]$.

By Theorem 14:

(a) For n odd, $[a_0, a_1, a_2, \ldots, a_n]$ is the a_nth term in the right branch of $[a_0, a_1, a_2, \ldots, a_{n-1}]$. That is, using Definition 16,

$$\begin{aligned} x_p &= [a_0, a_1, a_2, \ldots, a_{n-1}]\,, \\ &= [a_0, a_1, a_2, \ldots, a_{n-2}, a_{n-1}, a_n, 0]\,, \\ &= [a_0, a_1, a_2, \ldots, a_{n-2}, a_{n-1}, a_n, t] \end{aligned}$$

and x_s is the $(a_n - 1)$th term in the right branch of $[a_0, a_1, a_2, \ldots, a_{n-1}]$. That is,

$$\begin{aligned} x_s &= [a_0, a_1, a_2, \ldots, a_{n-1}, a_n - 1]\,, \\ &= [a_0, a_1, a_2, \ldots, a_{n-1}, a_n - 1, 1, 0]\,, \\ &= [a_0, a_1, a_2, \ldots, a_{n-1}, a_n - 1, 1, t]\,. \end{aligned}$$

By Definition 16, for the particular case $x = \frac{1}{0} = [0, 0]$, $x_p = [0, 0, 0] = [0] = \frac{0}{1}$ while x_s does not exist.

(b) For n even, $[a_0, a_1, a_2, \ldots, a_n]$ is the a_nth term in the left branch of $[a_0, a_1, a_2, \ldots, a_{n-1}]$. That is, using Definition 16,

$$\begin{aligned} x_s &= [a_0, a_1, a_2, \ldots, a_{n-1}]\,, \\ &= [a_0, a_1, a_2, \ldots, a_{n-2}, a_{n-1}, a_n, 0]\,, \\ &= [a_0, a_1, a_2, \ldots, a_{n-2}, a_{n-1}, a_n, t] \end{aligned}$$

and x_p is the $(a_n - 1)$th term in the left branch of $[a_0, a_1, a_2, \ldots, a_{n-1}]$. That is,

$$\begin{aligned} x_p &= [a_0, a_1, a_2, \ldots, a_{n-1}, a_n - 1]\,, \\ &= [a_0, a_1, a_2, \ldots, a_{n-1}, a_n - 1, 1, 0]\,, \\ &= [a_0, a_1, a_2, \ldots, a_{n-1}, a_n - 1, 1, t]\,. \end{aligned}$$

For $x = \frac{0}{1} = [0]$, $x_s = [0, 0] = \frac{1}{0}$ while x_p does not exist. □

12. Successors and predecessors in the Stern–Brocot tree

We now explore successors and predecessors in the Stern–Brocot tree.

Theorem 23. *If* $[a_0, a_1, a_2, \ldots, a_n]$ *is the short form of* x *where* $x \notin \left\{\frac{0}{1}, \frac{1}{1}, \frac{1}{0}\right\}$, *and* x_p, x, *and* x_s *are consecutive terms in a level of the Stern–Brocot tree, then, where* x_p *and* x_s *exist,*

(i) *For* n *odd,*

$$x_p = [a_0, a_1, a_2, \ldots, a_{n-2}, a_{n-1} - 1, 1, a_n] \quad \text{and}$$
$$x_s = [a_0, a_1, a_2, \ldots, a_{n-1}, a_n - 2, 2]\,.$$

(ii) *For* n *even,*

$$x_p = [a_0, a_1, a_2, \ldots, a_{n-1}, a_n - 2, 2] \quad \text{and}$$
$$x_s = [a_0, a_1, a_2, \ldots, a_{n-2}, a_{n-1} - 1, 1, a_n]\,.$$

Proof. There are two cases to consider:

(1) n odd:
Since $a_n > 1$, by Theorem 14, x is the first term in the left branch of $[a_0, a_1, a_2, \ldots, a_n - 1]$. Hence x_s is the first term in the right branch of $[a_0, a_1, a_2, \ldots, a_n - 1]$. That is, by Theorem 14 again,

$$x_s = [a_0, a_1, a_2, \ldots, a_{n-1}, a_n - 2, 2]\,.$$

x is also the a_nth term in the right branch of $[a_0, a_1, a_2, \ldots, a_{n-1}]$. Hence x_p is the a_nth term in the left branch of $[a_0, a_1, a_2, \ldots, a_{n-1}]$. That is, by Theorem 14,

$$x_p = [a_0, a_1, a_2, \ldots, a_{n-2}, a_{n-1} - 1, 1, a_n]\,.$$

For the particular case, $x = [0, a_1]$, $x_s = [0, a_1 - 2, 2]$ and x_p does not exist.

(2) n even:
Since $a_n > 1$, by Theorem 14, x is the first term in the right branch of $[a_0, a_1, a_2, \ldots, a_n - 1]$. Hence x_p is the first term in the left branch of $[a_0, a_1, a_2, \ldots, a_n - 1]$. That is, by Theorem 14 again,

$$x_p = [a_0, a_1, a_2, \ldots, a_{n-1}, a_n - 2, 2]\,.$$

x is also the a_nth term in the left branch of $[a_0, a_1, a_2, \ldots, a_{n-1}]$. Hence x_s is the a_nth term in the right branch of $[a_0, a_1, a_2, \ldots, a_{n-1}]$. That is, by Theorem 14,

$$x_s = [a_0, a_1, a_2, \ldots, a_{n-2}, a_{n-1} - 1, 1, a_n]\,.$$

For the particular case, $x = [a_0]$, $x_p = [a_0 - 2, 2]$ and x_s does not exist. □

13. Graphical properties of predecessors and successors of the Calkin–Wilf sequence

Some interesting patterns exist when we graph the fractional part of predecessors and successors of the Calkin–Wilf sequence. Recall the Gauss map:

Definition 17 (*Gauss Map*)**.** The *Gauss Map*, $G(x)$, is defined as

$$G(x) = \begin{cases} \left\{\frac{1}{x}\right\} & \text{for } x \in (0, 1] \\ 0 & \text{for } x = 0. \end{cases}$$

The Calkin–Wilf sequence is defined over $\mathbb{Q}$. We now define functions over the unit interval that are related to the successors and predecessors of the Calkin–Wilf sequence.

Definition 18 (*CW Successor and CW Predecessor Maps*)**.** Let the *CW Successor map*, $H(x)$, be defined as

$$H(x) = \frac{1}{\lfloor x \rfloor + 1 - \{x\}}$$

and the *CW Predecessor map*, $K(x)$, as

$$K(x) = \begin{cases} \dfrac{\frac{1}{x} - 1}{1}, & \text{for } \dfrac{1}{x} \in \mathbb{N} \\ \left\lfloor \dfrac{1}{x} \right\rfloor + 1 - \left\{ \dfrac{1}{x} \right\}, & \text{otherwise.} \end{cases}$$

It can be readily shown that $H \circ K(x) = K \circ H(x) = x$. Informally, the successor of the predecessor of x is x, and the predecessor of the successor of x is x.

Theorem 24. *For $x \in [0, 1)$, the graph of $y = \{H(x)\}$ is the rotation of $y = G(x)$ around $x = \frac{1}{2}$.*

Proof. For $x \in [0, 1)$,

$$\{H(x)\} = \left\{ \frac{1}{\lfloor x \rfloor + 1 - \{x\}} \right\} = \left\{ \frac{1}{1 - x} \right\} = G(1 - x). \quad \square$$

Theorem 25. *For $\left\{x \mid x \in (0, 1) \text{ and } x \neq \frac{1}{n}\right\}$, the graph of $y = \{K(x)\}$ is the rotation of $y = G(x)$ around $y = \frac{1}{2}$.*

Proof. For $x \in (0, 1)$ and $x \neq \frac{1}{n}$,

$$\{K(x)\} = \left\{ \left\lfloor \frac{1}{x} \right\rfloor + 1 - \left\{ \frac{1}{x} \right\} \right\} = 1 - \left\{ \frac{1}{x} \right\} = 1 - G(x). \quad \square$$

We have shown in [4] the following result:

Lemma 5. *For $x = [0, a_1, a_2, \ldots]$, $G^n(x) = [0, a_{n+1}, a_{n+2}, \ldots]$.*

Theorem 26. *For $t, n \in \mathbb{N}$,*

(i)

$$\{K \circ \{H(x)\}\} = \begin{cases} 1 - G(x) & \text{for } x \in \left(0, \dfrac{1}{2}\right) \\ 1 - G^3(x) & \text{for } x \in \left(\dfrac{1}{2}, 1\right) \text{ and } x \neq \dfrac{(t-1)\,n + 1}{tn + 1} \end{cases}$$

(ii)

$$\{H \circ \{K(x)\}\} = G^2(x) \quad \text{for } x \in (0, 1) \text{ and } x \neq \frac{1}{n}.$$

Proof. Let $x = [0, a_1, a_2, \ldots]$.

(1) From Theorem 25 and Lemma 4, for $\left\{x \mid x \in (0, 1) \text{ and } x \neq \frac{1}{n}\right\}$,

$$\begin{aligned} \{K(x)\} &= 1 - \left\{ \frac{1}{x} \right\} \\ &= [0, 1, a_2 - 1, a_3, a_4, \ldots] \end{aligned} \tag{13.1}$$

and from Theorem 24 and Lemmas 4 and 5, for $x \in [0, 1)$,

$$\{H(x)\} = \left\{ \frac{1}{1 - x} \right\}, \tag{13.2}$$

$$= \begin{cases} 0, & \text{for } x = 0 \\ [0, a_1 - 1, a_2, a_3, \ldots], & \text{for } a_1 > 1, \text{ that is, } x \in \left(0, \dfrac{1}{2}\right] \\ [0, a_3, a_4, \ldots], & \text{for } a_1 = 1, \text{ that is, } x \in \left(\dfrac{1}{2}, 1\right) \end{cases} \tag{13.3}$$

$\{K \circ \{H(x)\}\}$ is undefined whenever $\{H(x)\} = \frac{1}{n}$. Thus from (13.3), $\{K \circ \{H(x)\}\}$ is undefined whenever $\frac{1}{1-x} = t + \frac{1}{n}$, that is, whenever $x = \frac{(t-1)n+1}{tn+1}$. Similarly, $\{K \circ \{H(x)\}\}$ is undefined whenever $\{H(x)\} = 0$, that is, whenever $x = 0$. It follows that

(a) For $x \in \left(0, \frac{1}{2}\right)$,

$$\begin{aligned}\{K \circ \{H(x)\}\} &= \{K([0, a_1 - 1, a_2, a_3, \ldots])\},\\ &= [0, 1, a_2 - 1, a_3, a_4, \ldots],\\ &= 1 - G(x)\end{aligned}$$

$$\left\{K \circ \left\{H\left(\frac{1}{2}\right)\right\}\right\} = \{K(1)\} = 0 = 1 - G\left(\frac{1}{2}\right).$$

(b) For $x \in \left(\frac{1}{2}, 1\right)$ and $x \neq \frac{(t-1)n+1}{tn+1}$,

$$\begin{aligned}\{K \circ \{H(x)\}\} &= \{K([0, a_3, a_4, \ldots])\},\\ &= [0, 1, a_4 - 1, a_5, a_6, \ldots],\\ &= 1 - G^3(x).\end{aligned}$$

(2) For $x \in (0, 1)$ and $x \neq \frac{1}{n}$,

$$\{H \circ \{K(x)\}\} = \{H([0, 1, a_2 - 1, a_3, a_4, \ldots])\}.$$

From (13.1) there are two cases:

(a) $a_2 > 1$. From (13.3),

$$\begin{aligned}\{H \circ \{K(x)\}\} &= \{H([0, 1, a_2 - 1, a_3, a_4, \ldots])\},\\ &= [0, a_3, a_4, \ldots],\\ &= G^2(x).\end{aligned}$$

(b) $a_2 = 1$. From (13.3),

$$\begin{aligned}\{H \circ \{K(x)\}\} &= \{H([0, 1, 0, a_3, a_4, \ldots])\},\\ &= \{H([0, 1 + a_3, a_4, a_5, \ldots])\},\\ &= [0, a_3, a_4, \ldots],\\ &= G^2(x).\end{aligned}$$

Thus the two cases yield the same result, namely, $\{H \circ \{K(x)\}\} = G^2(x)$. □

Lemma 6. *For $x \in \left(0, \frac{1}{2}\right]$, $G^n(x) = G^{n+1}(1 - x)$.*

Proof. If $x \in \left(0, \frac{1}{2}\right]$, then $x = [0, a_1, a_2, \ldots]$ where $a_1 > 1$. By Lemmas 4 and 5,

$$\begin{aligned}G^{n+1}(1 - x) &= G^{n+1}([0, 1, a_1 - 1, a_2, a_3, \ldots]),\\ &= [0, a_{n+1}, a_{n+2}, \ldots],\\ &= G^n(x).\end{aligned}$$ □

Theorem 27. *For $x \in (0, 1)$ and $x \notin \left\{\frac{(t-1)n+1}{tn+1}, \frac{n}{n+1}\right\}$ where $t, n \in \mathbb{N}$,*

$$\{K \circ \{H(x)\}\} + \{H \circ \{K(1 - x)\}\} = 1.$$

Proof. By Theorem 26 and Lemma 6:

(i) For $x \in \left(0, \frac{1}{2}\right)$,

$$\begin{aligned}\{K \circ \{H(x)\}\} &= 1 - G(x),\\ &= 1 - G^2(1 - x),\\ &= 1 - \{H \circ \{K(1 - x)\}\}.\end{aligned}$$

(ii) For $x \in \left(\frac{1}{2}, 1\right)$ and $x \notin \left\{\frac{(t-1)n+1}{tn+1}, \frac{n}{n+1}\right\}$ where $t, n \in \mathbb{N}$,

$$\begin{aligned}\{K \circ \{H(x)\}\} &= 1 - G^3(x),\\ &= 1 - G^2(1 - x),\\ &= 1 - \{H \circ \{K(1 - x)\}\}.\end{aligned}$$ □

14. Invariant vertices

We investigate the conditions governing invariance between the Calkin–Wilf and Stern–Brocot trees.

Definition 19 (*Stern–Brocot Invariance*)**.** A vertex occupying the jth position in level n of the Calkin–Wilf tree is said to be Stern–Brocot invariant if it is also the term occupying the jth position in level n of the Stern–Brocot tree.

Theorem 28. *Let μ occupy the jth position in level n of the Calkin–Wilf tree where j has normalised additive factors $b_{j,0}, b_{j,1}, \ldots, b_{j,k}$. Then μ is Stern–Brocot invariant if and only if*

(1) *For $b_{j,0} < n-1$:*
   (a) $b_{j,k} = 0$, *and*
   (b) $b_{j,i} + b_{j,k-1-i} = n-1$ *where* $i = 0, 1, \ldots, \frac{k-2}{2}$.
(2) *For $b_{j,0} = n-1$:*

$$b_{j,i} + b_{j,k-1+i} = n-1 \quad \text{where } i = 1, 2, \ldots, \frac{k}{2}.$$

Proof. From Remark 3, Stern–Brocot invariance exists, if and only if,

$$\begin{aligned}&\left[b_{j,k}, \left(b_{j,k-1} - b_{j,k}\right), \left(b_{j,k-2} - b_{j,k-1}\right), \ldots, \left(b_{j,0} - b_{j,1}\right), \left(n - b_{j,0}\right)\right]\\ &\quad = \left[0, n - b_{j,0} - 1, \left(b_{j,0} - b_{j,1}\right), \left(b_{j,1} - b_{j,2}\right), \ldots, \left(b_{j,k-1} - b_{j,k}\right), b_{j,k} + 1\right]. \end{aligned} \tag{14.1}$$

(1) Since the first term on the right-hand side of (14.1) is 0, then $b_{j,k} = 0$, proving (a).
The left-hand side of (14.1) has $k+2$ terms while the right-hand side has $k+3$ terms. With $b_{j,k} = 0$, (14.1) becomes

$$\begin{aligned}&\left[0, b_{j,k-1}, \left(b_{j,k-2} - b_{j,k-1}\right), \ldots, \left(b_{j,0} - b_{j,1}\right), \left(n - b_{j,0}\right)\right]\\ &\quad = \left[0, n - b_{j,0} - 1, \left(b_{j,0} - b_{j,1}\right), \left(b_{j,1} - b_{j,2}\right), \ldots, \left(b_{j,k-1} + 1\right)\right] \end{aligned} \tag{14.2}$$

and has $k+2$ terms on both sides. Equating terms in (14.2) proves (b).
(2) Since $b_{j,0} = n-1$, (14.1) becomes

$$\begin{aligned}&\left[b_{j,k}, \left(b_{j,k-1} - b_{j,k}\right), \left(b_{j,k-2} - b_{j,k-1}\right), \ldots, \left(b_{j,1} - b_{j,2}\right), \left(b_{j,0} - b_{j,1} + 1\right)\right]\\ &\quad = \left[\left(b_{j,0} - b_{j,1}\right), \left(b_{j,1} - b_{j,2}\right), \left(b_{j,2} - b_{j,3}\right), \ldots, \left(b_{j,k-1} - b_{j,k}\right), b_{j,k} + 1\right] \end{aligned} \tag{14.3}$$

where both sides of (14.3) have $k+1$ terms. Equating terms in (14.3) proves (2). □

15. Connections with paperfolding

We conclude our investigation of the Calkin–Wilf tree by exploring a simple correspondence with the paperfolding sequence.

Take a sheet of paper and fold it once, right over left. When we open out this paper we see that we have a single crease. Now fold the paper twice. That is, fold it once as before and then fold it again, right over left. Upon opening the paper we see three creases. We observe that there are two different types of creases, one a V shape and the other a Λ shape. From here on we refer to the V crease as 1 and the Λ crease as 0. Hence after folding the paper twice and then unfolding we have the pattern: 110. We designate as $S_i = f_1 f_2 \ldots f_{2^i-1}$, the paperfolding sequence of creases produced after the ith folding and the infinite sequence $S = \lim_{i\to\infty} S_i$. Thus

$$\begin{aligned} S_1 &= 1,\\ S_2 &= 110,\\ S_3 &= 1101100,\\ S_4 &= 110110011100100. \end{aligned}$$

Definition 20 (*Interleave Operator*)**.** The interleave operator # acting on the two sequences $U = u_1u_2 \dots u_{n+1}$ and $V = v_1v_2 \dots v_n$ generates the following interleaved sequence:

$$U\#V = u_1v_1u_2v_2 \dots u_nv_nu_{n+1}.$$

Definition 21 (*Alternating Sequence*)**.** Let $A_{2r} = 1010 \cdots 10$ where the right-hand side is a sequence of length $2r$.

The following result, found in [7,11], tells us that the paperfolding sequence is obtained from a series of successive interleaves of alternating sequences based on the term 1.

Theorem 29. $S_i = A_{2^{i-1}}\#A_{2^{i-2}}\# \cdots \#A_2\#1$.

Theorem 30. *In the Calkin–Wilf tree, if the vertices in each level n* ($n > 0$) *that are greater than* 1 *are replaced by a* 0 *and all other vertices by a* 1 *,*

(i) *Level n* ($n > 1$) *becomes the alternating sequence,* $A_{2^{n-1}}$.
(ii) *When these altered levels* 1 *to n are interleaved, the paperfolding sequence* S_n *is obtained.*

Proof. (i) For level n ($n > 1$) of the Calkin–Wilf tree, each left child is less than 1 and each right child is greater than 1. The replacement gives the alternating sequence $A_{2^{n-1}}$.
(ii) The replacement gives 1 as the only term in level 1. Thus by (i) and Theorem 29, part (ii) is established. □

Acknowledgements

We thank the referees for their thorough reading of the paper which has resulted in improvements to its presentation.

References

[1] M. Aigner, G.M. Ziegler, Proofs from the Book, Springer, 2009.
[2] B.P. Bates, M.W. Bunder, K.P. Tognetti, Locating terms in the Stern–Brocot tree, European Journal of Combinatorics 31 (3) (2010) 1020–1033.
[3] B.P. Bates, M.W. Bunder, K.P. Tognetti, Linkages between the Gauss map and the Stern–Brocot tree, Acta Mathematica Academiae Paedagogicae Nyíregyháziensis 22 (3) (2006) 217–235.
[4] B.P. Bates, M.W. Bunder, K.P. Tognetti, Continued fractions and the Gauss map, Acta Mathematica Academiae Paedagogicae Nyíregyháziensis 21 (2005) 113–125.
[5] N. Calkin, H.S. Wilf, Recounting the rationals, American Mathematical Monthly 107 (4) (2000) 360–363.
[6] K.M. Courtright, J.A. Sellers, Arithmetic properties for hyper m-ary partition functions, Integers: Electronic Journal of Combinatorial Number Theory 4 (2004) #A06.
[7] C. Davis, D.E. Knuth, Number representations and Dragon curves — 1, Journal of Recreational Mathematics 3 (1970) 66–81.
[8] A.M. Hinz, S. Klavžar, U. Milutinović, D. Parisse, C. Petr, Metric properties of the tower of hanoi graphs and Stern's diatomic sequence, European Journal of Combinatorics 26 (2005) 693–708.
[9] S. Klavžar, U. Milutinović, C. Petr, Stern polynomials, Advances in Applied Mathematics 39 (1) (2007) 86–95.
[10] D.E. Knuth, Problem 10,906, American mathematical monthly; solution by Moshe Newman, American Mathematical Monthly 110 (7) (2003) 642–643.
[11] H. Prodinger, F.J. Urbanek, Infinite 0–1-sequences without long adjacent identical blocks, Discrete Mathematics 28 (1979) 277–289.
[12] N.J.A. Sloane, The on-line encyclopedia of integer sequences, 1999. Published electronically at www.research.att.com/~njas/sequences.