THE SUMMED PAPERFOLDING SEQUENCE

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ABSTRACT. The sequence $a(1), a(2), a(3), \ldots$, A088431 described in the Online Encyclopedia of Integer Sequences, is defined by: $a(n)$ is half of the $(n+1)$th component, that is, the $(n+2)$th term, of the continued fraction expansion of

$$\sum_{k=0}^{\infty} \frac{1}{2^{2^k}}.$$

Dimitri Hendriks has suggested that it is the sequence of run lengths of the paperfolding sequence, A014577. This paper proves several results of this summed paperfolding sequence and confirms Hendriks's conjecture.

This is a shortened form of the published paper and does not include references to Walnut software and continued fractions mentioned in the published version.

1. Introduction and Preliminaries

According to Ben-Abraham et al. [4], the regular paperfolding sequence was discovered in the mid-1960s by three NASA physicists: John Heighway, Bruce Banks and William Hartner. Martin Gardner celebrated their work in *Scientific American* [7], after which Davis and Knuth [5] developed the mathematical underpinnings of paperfolding. Since that time, many papers have been written exploring the diverse features of this sequence, notably Dekking et al. [6], Allouche and Shallit [1], Mendès France and van der Poorten [9] and Mendès France and Shallit [8].

We start with two alternative definitions of sequence A088431, each found at N.J.A. Sloane [10].

Definition 1. ***(A088431 Continued Fraction definition*** [10]***).*** *The sequence $A = a(1)\,a(2)\,a(3)\ldots$, A088431 of N.J.A. Sloane* [10]*, is defined by: $a(n)$ is half of the $(n+1)$th component, that is, the $(n+2)$th term, of the continued fraction expansion of*

$$\sum_{k=0}^{\infty} \frac{1}{2^{2^k}}.$$

Definition 2. ***(A088431 Alternative definition*** [10]***).*** *The sequence $A = a(1)\,a(2)\,a(3)\ldots$, A088431 of N.J.A. Sloane* [10]*, is given by the following rule: Let $i = 1, 2, 3, \cdots$ and $a(1) = 2$. Then*

$$a(a(1) + a(2) + a(3) + \cdots + a(n) + 1) = 2$$

and the ith undefined term of A is the ith term of the sequence

$$1, 3, 1, 3, 1, 3, 1, 3, 1, 3, 1, 3, \cdots$$

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Example 1. *Based on Definition 2:*

$a(n)$	*Formula*	*Value*
$a(1)$	$a(1) = 2$	2
$a(2)$	1, *since undefined*	1
$a(3)$	$a(a(1)+1) = a(2+1)$	2
$a(4)$	$a(a(1)+a(2)+1) = a(2+1+1)$	2
$a(5)$	3, *since undefined*	3
$a(6)$	$a(a(1)+a(2)+a(3)+1) = a(2+1+2+1)$	2
$a(7)$	1, *since undefined*	1

Dimitri Hendriks in Sloane [10] has suggested that sequence A088431 appears to be the sequence of run lengths of the regular paperfolding sequence A014577. In this paper we prove several results concerning this summed paperfolding sequence and confirm Hendriks's conjecture.

In what follows, for simplicity and where no ambiguity exists, we remove the commas from sequences. For example, for a sequence only having values $1, 2$ or 3, the sequence $2, 2, 1, 3$ becomes 2213.

Davis and Knuth [5] prove the following result which we adopt as a definition for the paperfolding sequence. The notation is taken from Bates et al. [2].

Definition 3. ***(Paperfolding sequence)*** *Let S_n be the paperfolding sequence of length $2^n - 1$. Let also $\overline{S_n^R}$ be S_n in reverse order with $0s$ becoming $1s$, and $1s$ becoming $0s$. Then, for $S_1 = 1$,*

$$\begin{aligned} S_{n+1} &= S_n 1 \overline{S_n^R} \text{ and} \\ \overline{S_{n+1}^R} &= S_n 0 \overline{S_n^R}. \end{aligned}$$

Bates et al. [2] and [3] identify the following results for S_n:

Theorem 1. ***(Expression for S_n)*** *For $n > 0$ and $h, k \geq 0$, $S_n = f_1 f_2 \ldots f_{2^n-1}$ where,*

$$f_i = \begin{cases} 1, & \text{if } i = 2^k(4h+1) \\ 0, & \text{if } i = 2^k(4h+3). \end{cases}$$

Theorem 2. ***(Paperfolding runs)*** *The paperfolding sequence, S_n, contains only runs of single, double or triple entries of 0; or single, double or triple entries of 1. In particular, for $n \geq 4$, S_n contains*

i) 2^{n-4} instances of the triple, 111, and $2^{n-4} - 1$ instances of the triple 000;
ii) 2^{n-3} instances of the double, 11, and $2^{n-3} + 1$ instances of the double 00;
iii) 2^{n-4} instances each of the single 1, and the single 0.

Theorem 3. ***(Interleaved paperfolding)*** *Let $S = f_1 f_2 f_3 \ldots$ be the paperfolding sequence of infinite length. Then*

$$S = S_3 \ f_1 \ \overline{S_3^R} \ f_2 \ S_3 \ f_3 \ \overline{S_3^R} \ f_4 \ S_3 \ f_5 \ \overline{S_3^R} \ \ldots$$

That is, S is formed from the alternating interleave of two subsequences, S_3 and $\overline{S_3^R}$, with itself.

We consider the *runs* of identical terms in S_n or S, that is, the sizes of the sequence of successive $1s$ and $0s$ of S_n or S. We begin with a definition of these runs.

Definition 4. ***(Summed paperfolding sequence)*** *For $S_n = f_1 f_2 \ldots f_{2^n-1}$,*

- *The summed paperfolding sequence, G_n, is the sequence of the sizes of successive* 1*s and* 0*s of S_n.*
- *The summed paperfolding sequence of infinite length is $G = \lim_{n\to\infty} G_n$; and is designated as $G = g(1)\, g(2)\, g(3) \ldots$*

We show in Theorem 4 that G_n has length 2^{n-1}.

Example 2.

$$\begin{aligned} S_4 &= 110110011100100. \\ G_4 &= 21223212 \\ G &= 21223212\ldots \end{aligned}$$

The key results in this paper are:

- A closed-form expression for G is developed (Theorem 5) analogous to the expression for S_n (and by extension S) given at Theorem 1.
- The main internal relationships within G are identified. (Theorem 6)
- **Main Result:** Confirmation of Hendriks's conjecture [10] is given (Theorem 9). That is, the sequence $A = a(1)\, a(2)\, a(3) \ldots$ of A088431 is exactly the sequence $G = g(1)\, g(2)\, g(3) \ldots$of Definition 4.

Theorem 4. ***(Length of** G_n**).*** $|G_n| = 2^{n-1}$, *where $|G_n|$ is the length of G_n.*

Proof. For $n < 4$ our result is true. From Theorem 2, for $n \geq 4$, there are 2^{n-4}, $111s$; $2^{n-4}-1$, $000s$; 2^{n-3}, $11s$; $2^{n-3}+1$, $00s$; 2^{n-4}, $1s$; and 2^{n-4} $0s$. Hence

$$\begin{aligned} |G_n| &= 2^{n-4} + 2^{n-4} - 1 + 2^{n-3} + 2^{n-3} + 1 + 2^{n-4} + 2^{n-4} \\ &= 2^{n-1}. \end{aligned}$$

□

Note that from Definition 3, since $S_{n+1} = S_n 1 \overline{S_n^R}$, the initial $|G_n|$ terms of G_{n+1}, will be G_n. Since by Theorem 4, $|G_n| = 2^{n-1}$, we can write

$$G_n = g(1)\, g(2) \ldots g\left(2^{n-1}-1\right) g\left(2^{n-1}\right).$$

2. Basic facts of G

We now develop an analogous expression for G to that given at Theorem 1 for S_n (and, by extension, S).

Theorem 5. ***(Expression for*** G***)*** *For* $h,k,p \geq 0$, $G = g(1)\,g(2)\,g(3)\dots$ *where,*

$$g(n) = \begin{cases} 1, & \text{if i) } n = 8k+2, \text{ or} \\ & \text{if ii) } n = 8k+7. \\ \\ 2, & \text{if iii) } n = 8k+1 \text{ and } k = 2^p(4h+3), \text{ or} \\ & \text{if iv) } n = 8k+3, \text{ or} \\ & \text{if v) } n = 8k+4 \text{ and } k \text{ is even, or} \\ & \text{if vi) } n = 8k+5 \text{ and } k \text{ is odd, or} \\ & \text{if vii) } n = 8k+6, \text{ or} \\ & \text{if viii) } n = 8k+8 \text{ and } k+1 = 2^p(4h+1). \\ \\ 3, & \text{if ix) } n = 8k+1 \text{ and } k = 2^p(4h+1), \text{ or} \\ & \text{if x) } n = 8k+4 \text{ and } k \text{ is odd, or} \\ & \text{if xi) } n = 8k+5 \text{ and } k \text{ is even, or} \\ & \text{if xii) } n = 8k+8 \text{ and } k+1 = 2^p(4h+3). \end{cases}$$

Proof. From Theorem 2, S only contains singles, doubles and triples. Thus the only possible terms in G are 1, 2 and 3. From Theorem 3, each S_3 and $\overline{S_3^R}$ starts with 11 and ends with 00 and the component $S_3\ f_i\ \overline{S_3^R}\ f_{i+1}$ is followed by $S_3\ f_{i+2}\ \overline{S_3^R}\ f_{i+3}$, which is followed by $S_3\ f_{i+4}\ \overline{S_3^R}\ f_{i+5}$, and so on, indefinitely. Let the 0th component be $S_3\ f_1\ \overline{S_3^R}\ f_2$. Then the kth component is $S_3\ f_i\ \overline{S_3^R}\ f_{i+1}$ where i is odd and $k = \frac{i-1}{2}$. Thus the translation from S to G of the kth component, $S_3\ f_i\ \overline{S_3^R}\ f_{i+1}$, can be represented by 8 terms, $g(8k+1)$ to $g(8k+8)$, where $k = \frac{i-1}{2}$ with two possible configurations:

- $f_i = 0 : S_3\ f_i\ \overline{S_3^R}\ f_{i+1}$ becomes (2 or 3)123221(2 or 3), or
- $f_i = 1 : S_3\ f_i\ \overline{S_3^R}\ f_{i+1}$ becomes (2 or 3)122321(2 or 3)

where bracketed entries are determined by the values of f_{i-1} and f_{i+1}.

We consider each of $g(n) = 1$, 3 and 2 separately:

1. $g(n) = 1$. In the 8-term translations above, we have $g(8k+2)$ and $g(8k+7)$ *always* taking the value 1, irrespective of the values of f_{i-1}, f_i or f_{i+1}, and there are *no other* values of 1 in this 8-term translation. Thus $g(n) = 1$ if $n = 8k+2$ or $8k+7$.

2. $g(n) = 3$. Consider the component $S_3\ f_i\ \overline{S_3^R}\ f_{i+1}$:

i) If $f_i = 0$, then $g(4i) = g(8k+4)$ where from Theorem 1, $i = 2^p(4h+3)$. Since i is odd, $p = 0$. Thus $8k+4 = 4(4h+3)$ and so, $k = 2h+1$ which is odd. It follows that $g(8k+4) = 3$, for k odd.

ii) If $f_{i+1} = 0$, then $g(4(i+1)) = g(8k+8)$ where from Theorem 1, $i + 1 = 2^m(4h+3)$. Since $i+1$ is even, $m > 0$. Thus $8k+8 = 4(i+1) = 2^{m+2}(4h+3)$. That is, $k+1 = 2^{m-1}(4h+3) = 2^p(4h+3)$ for $p = m-1$, that is, for $p \geq 0$. It follows that $g(8k+8) = 3$, for $k+1 = 2^p(4h+3)$ where $p \geq 0$.

iii) If $f_i = 1$, then $g(4i+1) = g(8k+5)$ where from Theorem 1, $i = 2^p(4h+1)$. Since i is odd, $p = 0$. Thus $8k+5 = 4(4h+1)+1$ and so, $k = 2h$. It follows that $g(8k+5) = 3$, for k even.

iv) If $f_{i-1} = 1$, then for $i \geq 3$, that is, $k \geq 1$, $g(4(i-1)+1)) = g(4i-3) = g(8k+1)$ where from Theorem 1, $i - 1 = 2^m(4h+1)$. Since $i-1$ is even,

$m>0$. Thus $8k+1=2^{m+2}(4h+1)+1$ that is, $k=2^{m-1}(4h+1)$. It follows that $g(8k+1)=3$, if $k=2^{p}(4h+1)$ for $p=m-1\geq 0$.

3. $g(n)=2$. Since the only possible terms in G are 1, 2 and 3, all terms not part of i) and ii) must be terms having value 2. The result follows. □

The following theorem identifies important internal relationships within G.

Theorem 6. *(**Relationships in** G). Let $G=g(1)\,g(2)\,g(3)\ldots$ be the summed paperfolding sequence of infinite length, then:*

a) $g(2)=1$; $g(2^{n})=2$, *for* $n>1$.
b) $g(3)=2$; $g(2^{n}+1)=3$, *for* $n>1$.
c) $g(2^{n}-i)=g(i+1)$, *for* $0\leq i<2^{n-1}-1$.
d) $g(2^{n}+i)=g(i)$, *for* $1<i<2^{n-1}$ *or* $2^{n-1}+1<i<2^{n}-1$.
e) $g(6)=2$; $g\left(2^{n}+2^{n-1}\right)=3$, *for* $n>2$.
f) $g(7)=1$; $g\left(2^{n}+2^{n-1}+1\right)=2$, *for* $n>2$.
g) $g(2^{n}+2^{m})=g(2^{m})=2$, *for* $n>m+1>2$.
h) $g(2^{n}+2^{m}+2^{r})=g(2^{m}+2^{r})$, *for* $n>m+1>r+1$.
i) $g\left(2^{k_1}+2^{k_2}+\ldots+2^{k_r}\right)=g\left(2^{k_2}+2^{k_3}+\ldots+2^{k_r}\right)$, *for* $k_1>k_2>\ldots>k_r$ *and* $r>2$.
j) $g\left(2^{k_1}+2^{k_2}+\ldots+2^{k_r}\right)=g\left(2^{k_{r-2}}+2^{k_{r-1}}+2^{k_r}\right)$, *for* $k_1>k_2>\ldots>k_r$ *and* $r>2$.

Proof. By Theorem 5, with its subcases denoted by i) to xii) :

a) $g(2)=1$ by i) ; $g(4)=2$ by v); $g(2^{n})=2$, for $n>2$ by viii).

b) $g(3)=2$ by iv); $g(5)=3$ by xi); $g(2^{n}+1)=3$, for $n>2$ by ix).

c) The first $2^{n-1}-1$ elements of G_{n+1} are the sums of runs of 1s and 0s and the last $2^{n-1}-1$ elements are the same sums, but of 0s and 1s and in reverse order. So, for $0\leq i<2^{n-1}-1$,

$$g(2^{n}-i)=g(i+1).$$

d) As $2^{n}+i=2^{n+1}-(2^{n}-i)$, if $1<i<2^{n-1}$, by c), applied twice

$$g(2^{n}+i)=g(2^{n}-(i-1))=g(i).$$

If $2^{n-1}+1<i<2^{n}-1$, then $i=2^{n-1}+2^{n-2}+\ldots+2^{r}+j$, where either $r=n-1$ and $1<j<2^{r-1}-1$ or $r<n-1$ and $0\leq j<2^{r-1}-1$. In both cases, by c),

$$\begin{aligned}g(2^{n}+i)&=g\left(2^{n+1}-(2^{r}-j)\right)\\&=g(2^{r}-(j-1))\\&=g(2^{n}-(2^{r}-j))\\&=g(i).\end{aligned}$$

e) $g(6)=2$ by vii); $g(12)=3$ by x); $g\left(2^{n}+2^{n-1}\right)=g\left(3.2^{n-1}\right)=3$, for $n>3$ and $2^{n-1}+1<i<2^{n}-1$, by xii).

f) $g(7)=1$ by ii), $g(13)=2$ by vi); $g\left(2^{n}+2^{n-1}+1\right)=g\left(3.2^{n-1}+1\right)=2$, for $n>3$ by iii).

g) For $p=n-m\geq 2$, $g(2^{n}+2^{m})=g(2^{m}(2^{p}+1))=2$ by viii) and $g(2^{m})=2$ if $m=0,1$ and $g(2^{m})=2$ if $m\geq 3$, by viii). Thus if $n>m+1>2$, $g(2^{n}+2^{m})=g(2^{m})=2$, by d) and a).

h) For $n>m+1$, or $r>0$,

$$\begin{aligned}g(2^{m}+2^{r})&=g\left(2^{r}\left(2^{m-r}+1\right)\right)\\&=g(2^{r}(4h_0+1))=2\text{ by viii); and}\end{aligned}$$

For $n > m + 1 > r + 1$, by d),

$$\begin{aligned} g\left(2^n + 2^m + 2^r\right) &= g\left(2^r\left(2^{n-r} + (4h_0 + 1)\right)\right) \\ &= g\left(2^r\left(4h_1 + (4h_0 + 1)\right)\right) \\ &= g\left(2^r\left(4\left(h_1 + h_0\right) + 1\right)\right) = 2 \text{ by viii}). \end{aligned}$$

i) For $r > 2$ and $k_1 > k_2 > \ldots > k_r$, by d),

$$g\left(2^{k_1} + 2^{k_2} + \ldots + 2^{k_r}\right) = g\left(2^{k_2} + 2^{k_3} + \ldots + 2^{k_r}\right).$$

j) By repeated use of part i) of this proof. □

Note that for $n > 1$, the term at $g(2^n) = 2$ and for $n > 2$, $g(2^n) + 1 = 3$. This follows from observing that the sequence prior to $g(2^{n-1})$ is mirrored to give the sequence after $g(2^{n-1} + 1)$, reflected around $2, 3$ in each case.

3. The expression $h(n)$

The following definition is important in developing our main result.

Definition 5. ***(Expression for*** $h(n)$***).*** *For* $n > 0$,

$$h(n) = g(1) + g(2) + g(3) + \ldots + g(n).$$

Theorem 7. ***(Relationships involving*** $h(n)$***).*** *For* $n > 0$,

a) If $n = 4q + 1, h(n) = 2n$.
b) If $n = 2^k(4q + 1)$ and $k > 0, h(n) = 2n - 1$.
c) If $n = 4q + 3, h(n) = 2n - 1$.
d) If $n = 2^k(4q + 3)$ and $k > 0, h(n) = 2n$.

Proof. By induction on n. For $n = 1, 5$ and 6, $h(n) = 2n$; and for $n = 2, 3$ and 4, $h(n) = 2n - 1$. So *a)* to *d)* hold for minimal values of n. Assume *a)* to *d)* for values less than some n. If $n = 2^k(4q + 1) > 3$, let $q = \sum_{i=1}^{r} 2^{q_i}$. So

$$n = 2^k + \sum_{i=1}^{r} 2^{q_i+k+2}$$

and

$$h(n) = \sum_{j=1}^{2^{q_1+k+2}} g(j) + \sum_{j=1}^{n-2^{q_1+k+2}} g\left(2^{q_1+k+2} + j\right).$$

If $q_2 < q_1 - 1, n - 2^{q_1+k+2} < 2^{q_1+k+1}$, so by Theorem 6, *b)* and *d)*,

$$\begin{aligned} g\left(2^{q_1+k+2} + 1\right) &= 3 = g(1) + 1, \text{ and} \\ h(n) &= \sum_{j=1}^{2^{q_1+k+2}} g(j) + 1 + \sum_{j=1}^{n-2^{q_1+k+2}} g(j) \\ &= h\left(2^{q_1+k+2}\right) + 1 + h\left(n - 2^{q_1+k+2}\right). \end{aligned}$$

So by the induction hypothesis, if $k > 0$,

$$\begin{aligned} h(n) &= 2^{q_1+k+3} - 1 + 1 + 2n - 2^{q_1+k+3} - 1 \\ &= 2n - 1. \end{aligned}$$

If $q_2 = q_1 - 1$, by Theorem 6, $e), b)$ and $f)$,

$$\begin{aligned} g\left(2^{q_1+k+2} + 2^{q_2+k+2}\right) &= 3 = g\left(2^{q_2+k+2}\right) + 1, \text{ and} \\ g\left(2^{q_1+k+2} + 2^{q_2+k+2} + 1\right) &= 2 = g\left(2^{q_2+k+2} + 1\right) - 1. \end{aligned}$$

So by Theorem 6, $d)$, for the other values of j,

$$\begin{aligned} h(n) &= \sum_{j=1}^{2^{q_1+k+2}} g(j) + 1 + \sum_{j=1}^{n-2^{q_1+k+2}} g(j) \\ &= 2n - 1, \text{ as before, so we have } b). \end{aligned}$$

If, instead, $k = 0$, the induction hypothesis gives

$$\begin{aligned} h(n) &= 2^{q_1+k+3} - 1 + 1 + 2n - 2^{q_1+k+3} \\ &= 2n, \text{ so we have } a). \end{aligned}$$

If $n = 2^k(4q+3)$,the working is as above, except that the induction hypothesis gives, for $k > 0$,

$$\begin{aligned} h(n) &= 2^{q_1+k+3} - 1 + 1 + 2n - 2^{q_1+k+3} \\ &= 2n, \end{aligned}$$

and for $k = 0$,

$$\begin{aligned} h(n) &= 2^{q_1+k+3} - 1 + 1 + 2n - 2^{q_1+k+3} - 1 \\ &= 2n - 1, \text{ so we have } d) \text{ and } c). \end{aligned}$$

□

Theorem 8. ***(Limits on*** $h(n)$***).*** *If* $h(n) + 1 \neq 7 \bmod 8$ *and* $\neq 2 \bmod 8$, *then*

$$g(h(n) + 1) = 2$$

and for no other values of m *is* $g(m) = 2$.

Proof. If $n = 4q + 1$, by Theorem 7 $a)$,

$$h(n) + 1 = 2n + 1 = 8q + 3.$$

So by Theorem 5, $iv)$ we have the result.

If $n = 2^k(4q+1)$ and $k > 0$, by Theorem 7 $b)$,

$$h(n) + 1 = 2n = 2^{k+1}(4q+1)$$

and we have the result by Theorem 5, $viii)$ if $k > 1$, and by Theorem 5, $v)$ if $k = 1$.

If $n = 4q + 3$, by Theorem 7 $c)$,

$$h(n) + 1 = 2n = 8q + 6.$$

So by Theorem 5, $vii)$, we have the result.

If $n = 2^k(4q+3)$, by Theorem 7 $d)$,

$$h(n) + 1 = 2n + 1 = 2^{k+3}q + 2^{k+2} + 2^{k+1} + 1.$$

So by Theorem 6, $j)$, we have

$$g(h(n) + 1) = g\left(2^{k+2} + 2^{k+1} + 1\right).$$

So by Theorem 6, $f)$ we have the result.

So we have $g(m) = 2$, where $m = 8q + 3$, $8q + 6$, $2^{k+1}(4q+1)$ with $k > 0$ and $2^{k+1}(4q+3) + 1$ with $k > 0$.

By Theorem 7, the value of $g(m)$ when $m = 8q+7$ or $8q+2$ is 1 and for $m = 8q+1$, $2^{k+1}(4q+3)$ with $k > 0$ or $2^{k+1}(4q+1)+1$ with $k > 0$, $g(m) = 3$.

So all the cases when $g(m) = 2$ are obtained when $m = h(n)+1$. □

4. Confirmation of Hendriks's conjecture

We are now able to state our main result, namely, that the sequence $A = a(1)\,a(2)\,a(3)\ldots$ of A088431 is exactly the sequence $G = g(1)\,g(2)\,g(3)\ldots$ This was conjectured by Dimitri Hendriks in Sloane [10].

Theorem 9. ***(Confirmation of Hendriks's conjecture)*** *$a(n) = g(n)$, that is, the sequences A and G are the same.*

Proof. By Theorem 6 c),

$$g(2^n - i) = g(i)$$

for $0 \leq i < 2^{n-1} - 1$, and

$$g\left(2^n - 2^{n-1} + 1\right) = g\left(2^{n-1} - 1\right) = 3 = g\left(2^{n-1}\right) + 1.$$

If we let G_n^{R+1} be the reverse of G_n with a 1 added to the new first term, we have

$$G_{n+1} = G_n G_n^{R+1}.$$

Since

$$G_5 = 2122321231232212,$$

G_5 has the subsequences: $321, 123, 1223, 3221, 212, 312, 232$ and 231. Thus 321 and 213 will appear in G_5^{R+1} and so in G_6. As for $n > 3$,

$$\begin{aligned} g\left(2^{n-1}\right) &= g\left(2^{n-1}+2\right) = 2, \text{ and} \\ g\left(2^{n-1}+1\right) &= 3, \end{aligned}$$

the middle sequence of any G_n will be 232. No new sequence of this kind can be generated in any G_n or G_n^{R+1}. Leaving out all the 2s in G, every 1 is followed by a 3 and every 3 is followed by a 1, giving $1, 3, 1, 3, \ldots$. By Theorem 8, $g(n)$ has the defining properties of $a(n)$ given at Definition 2. □

References

[1] J.-P Allouche, J. Shallit, *Automatic Sequences. Theory, Applications, Generalizations, Cambridge University Press,* Cambridge, United Kingdom, 2003.

[2] B.P. Bates, M.W. Bunder, K.P. Tognetti, Mirroring and Interleaving in the Paperfolding sequence, *Appl. Anal. Discrete Math* **4** (2010) 96–118.

[3] B.P. Bates, M.W. Bunder, K.P. Tognetti, Self-Matching Bands in the Paperfolding sequence. *Appl. Anal. Discrete Math* **5** (2011) 46–54.

[4] S.I. Ben-Abraham, A Quandt, D. Shapira, Multidimensional paperfolding systems, *Acta Cryst.* (2013). **A69** 123–130.

[5] C. Davis, D.E. Knuth, Number Representations and Dragon Curves*, Journal of Recreational Mathematics* **3** (1970) Part 1: 66–81, Part 2: 133–149.

[6] F. M. Dekking, M. Mendes France, A.J. van der Poorten, Folds!*, The Mathematical Intelligencer* **4**: 130–138, II: ibid. 173–181, III: ibid. 190–195, 1982.

[7] M. Gardner, Mathematical Games*, Sci. Am* **216 No. 3** (March 1967) 124–125; **216 No. 4** (April 1967) 118–120**; 217 No. 1** (July 1967) 115.

[8] M. Mendès France, J.O. Shallit, Wire Bending, *Journal of Combinatorial Theory, Series A*, **50**: 1–23, 1989.

[9] M. Mendès France, A.J. van der Poorten, Arithmetic and Analytical Properties of Paper Folding Sequences, *Bulletin of the Australian Mathematical Society*, 123–131 (1981).

[10] N.J.A. Sloane, Online Encyclopedia of Integer Sequences, https://oeis.org/A088431 [accessed 8th February 2024]

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