

## Applications of Calculus: Simple Harmonic Motion

**Time required**  
45 minutes

ID: **XXXX**

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### Activity Overview

In this activity, students are introduced to the important topic of simple harmonic motion in terms of the motion of a child on a swing. Using multiple representations to support understanding, students derive the defining formulas – first, beginning with the trigonometric relationship between time and displacement, and differentiating up to the form for acceleration, then by integration from acceleration back to displacement. This activity provides an introduction to differential equations.

### Concepts

- Differentiation and Integration of standard forms, substitution methods for integration.
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### Teacher Preparation

This investigation offers opportunities for review and consolidation of key concepts related to differentiation and integration of trigonometric functions. It provides a firm link between the theory and applications of the Calculus. As such, care should be taken to provide ample time for ALL students to engage actively with the requirements of the task, allowing some who may have missed aspects of earlier work the opportunity to build new and deeper understanding.

- This activity can serve to consolidate earlier work on differentiation and integration. It offers a suitable introduction to differential equations.
- Begin by reviewing the method of differentiation of trigonometric functions, and methods of integration of the standard function forms.
- The screenshots on pages X–X (top) demonstrate expected student results. Refer to the screenshots on page X (bottom) for a preview of the student .tns file.
- **To download the .tns file, go to <http://education.ti.com/exchange> and enter “XXXX” in the search box.**

### Classroom Management

- This activity is intended to be **teacher led**. You should seat your students in pairs so they can work cooperatively on their handhelds. Use the following pages to present the material to the class and encourage discussion. Students will follow along using their handhelds, although the majority of the ideas and concepts are only presented in **this** document; be sure to cover all the material necessary for students' total comprehension.
- Students can either record their answers on the handheld or you may wish to have the class record their answers on a separate sheet of paper.

### TI-Nspire™ Applications

Graphs & Geometry, Calculator, Notes.

**Step 1:** Begin with discussion and review of both the differentiation and integration of standard forms and of the trigonometric functions in particular. Ensure that students are comfortable with these and then challenge them to apply what they know about these processes to real-world situations, with a particular focus upon rates of change.

1.1 1.2 1.3 1.4 ▶RAD AUTO REAL

Question

1. Study the motion of the particle P above. Carefully describe the motion of this particle over time.

Answer

**Step 2:** Motion on a swing should be something that all students can relate to. It offers a suitable context for a closer examination of rates of change of displacement leading to velocity, and rate of change of velocity leading to acceleration.

1.2 1.3 1.4 1.5 ▶RAD AUTO REAL

Now imagine that the point P represents your motion on a swing.

Sit on the swing, push back to the point A and then - swing!

Picture yourself moving back and forth... and then answer these questions.

**Step 3:** Particular care should be taken to build understanding of this concept of acceleration – while students will relate readily to displacement and velocity, acceleration is best introduced in terms of force (being careful to distinguish between them!) Once established, this forms the basis for deriving the forms for simple harmonic motion, beginning with the standard relationship between time and displacement, leading to velocity and finally to acceleration. Substituting back leads readily to the defining equation in terms of acceleration and displacement.

1.8 1.9 1.10 1.11 ▶RAD AUTO REAL

Motion on a swing is an example of **simple harmonic motion**. Motion is defined to be simple harmonic if the acceleration is directly proportional to displacement from the origin.

i.e.  $a = \frac{d^2}{dt^2}(x) = kx$

**Step 4:** The second part of this activity offers a series of extensions, beginning with the challenge for students to describe other examples of simple harmonic motion. They should then be introduced to the variety of forms for acceleration, and attempt to justify these from their knowledge of rates of change.

TI-Nspire calculator interface showing question EX2 and its answer. The interface includes a navigation bar with tabs 1.17, 2.1, 2.2, and 2.3, and a mode selector set to RAD, AUTO, and REAL. The question text is: "EX2. Carefully study the following: explain and justify each." Below the text is the equation:  $a = \frac{d^2}{dt^2}(x) = \frac{dv}{dt}$  and  $a = v \cdot \frac{dv}{dx}$ . The answer section is currently empty, indicated by a downward-pointing arrow.

**Step 5:** What follows supports students in deriving the simple harmonic forms by integrating from the acceleration equation. This involves some substitution of critical values drawn from the physical example of the child on the swing and, finally, integration of inverse trigonometric function forms. When deriving this form, students should be encouraged to discuss the use of the sine or the cosine form – each appropriate depending upon the physical conditions. This should lead to consideration of the **phase shift** as related to the starting point of the motion.

TI-Nspire calculator interface showing the derivation of the velocity equation. The interface includes a navigation bar with tabs 2.3, 2.4, 2.5, and 2.6, and a mode selector set to RAD, AUTO, and REAL. The text reads: "We can use these results to derive the equations for Simple Harmonic Motion beginning with the defining relationship  $a = -n^2 \cdot x$ . Integrating both sides:" followed by the equation  $\frac{1}{2}v^2 = C - \frac{1}{2}n^2 \cdot x^2$ .

TI-Nspire calculator interface showing question EX4 and its answer. The interface includes a navigation bar with tabs 2.5, 2.6, 2.7, and 2.8, and a mode selector set to RAD, AUTO, and REAL. The question text is: "EX4. When might it be appropriate to express this equation using cosine instead of sine?" The answer section is currently empty, indicated by a downward-pointing arrow.

**Applications of Calculus: Simple Harmonic Motion – ID: XXXX**

(Student)TI-Nspire File: *CalcActXX\_Simple\_Harmonic\_Motion\_EN.tns*

1.1 1.2 1.3 1.4 ▸RAD AUTO REAL

Applications of Calculus  
Simple Harmonic Motion

Calculus

1.1 1.2 1.3 1.4 ▸RAD AUTO REAL



Question

1. Study the motion of the particle P above. Carefully describe the motion of this particle over time.

Answer

1.1 1.2 1.3 1.4 ▸RAD AUTO REAL

Question

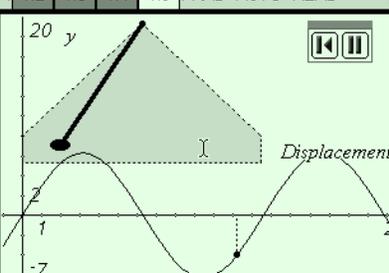
2. What general function could be given to describe the position  $x$  of P at any time  $t$  seconds?

Answer

1.1 1.2 1.3 1.4 ▸RAD AUTO REAL

Now imagine that the point P represents your motion on a swing.  
Sit on the swing, push back to the point A and then – swing!  
Picture yourself moving back and forth... and then answer these questions.

1.2 1.3 1.4 1.5 ▸RAD AUTO REAL



Displacement

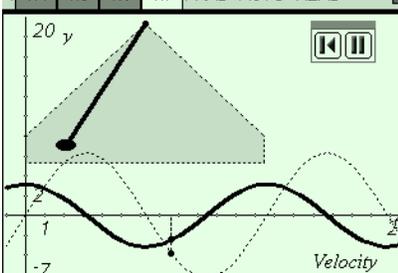
1.3 1.4 1.5 1.6 ▸RAD AUTO REAL

Question

3. Focus first upon your speed: when are you moving fastest? When do you stop?

Answer

1.4 1.5 1.6 1.7 ▸RAD AUTO REAL



Velocity

1.5 1.6 1.7 1.8 ▸RAD AUTO REAL

Question

4. Now focus on the force acting upon you – once you start swinging, there is only one force acting: gravity, pulling downwards. When is this action greatest? When least?

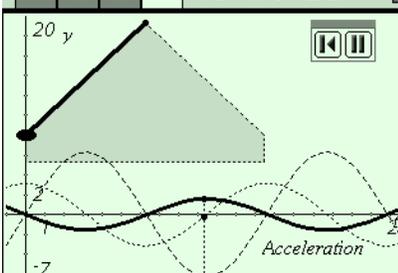
Answer

1.9 1.10 1.11 1.12 ▸RAD AUTO REAL

By Newton's Laws of Motion, force is directly proportional to acceleration – in fact,  $F = m \cdot a$ , where  $m$  is your mass, and  $a$  your acceleration.

Since your mass is constant, the force you experience when on the swing is proportionally equivalent to your acceleration.

1.7 1.8 1.9 1.10 ▸RAD AUTO REAL



Acceleration

1.8 1.9 1.10 1.11 ▸RAD AUTO REAL

Motion on a swing is an example of **simple harmonic motion**. Motion is defined to be simple harmonic if the acceleration is directly proportional to displacement from the origin.

i.e.  $a = \frac{d^2}{dt^2}(x) = k \cdot x$

1.9 1.10 1.11 1.12 ▸RAD AUTO REAL

Question

5. Clearly explain what this statement means.

Answer

1.10 1.11 1.12 1.13 ▸ RAD AUTO REAL

**Question**

6. Use the displacement equation to derive a formula for velocity.

**Answer** ▾

1.11 1.12 1.13 1.14 ▸ RAD AUTO REAL

**Question**

7. From this equation, derive a formula for acceleration in relation to time.

**Answer** ▾

1.12 1.13 1.14 1.15 ▸ RAD AUTO REAL

**Question**

8. Now substitute the formula for displacement into this acceleration formula.

**Answer** ▾

1.13 1.14 1.15 1.16 ▸ RAD AUTO REAL

**Question**

9. In your own words, clearly describe how this formula relates to our motion on a swing.

**Answer** ▾

1.14 1.15 1.16 1.17 ▸ RAD AUTO REAL

**Question**

10. Referring to the graphs of motion you have just seen, carefully describe the critical points of this motion in terms of displacement, velocity and acceleration.

**Answer** ▾

1.15 1.16 1.17 2.1 ▸ RAD AUTO REAL

**Simple Harmonic Motion**

**Extension Activities**

1.16 1.17 2.1 2.2 ▸ RAD AUTO REAL

**Question**

**EX1.** Find other examples of simple harmonic motion, and try to analyze these in the same way as we have for the child on the swing.

**Answer** ▾

1.17 2.1 2.2 2.3 ▸ RAD AUTO REAL

**Question**

**EX2.** Carefully study the following: explain and justify each.

$$a = \frac{d^2}{dt^2}(x) = \frac{dv}{dt} \text{ and } a = v \cdot \frac{dv}{dx}$$

**Answer** ▾

2.1 2.2 2.3 2.4 ▸ RAD AUTO REAL

**Question**

**EX3.** Show that  $\int v \cdot \frac{dv}{dx} dx = \frac{1}{2} v^2$

**Answer** ▾

2.2 2.3 2.4 2.5 ▸ RAD AUTO REAL

We can use these results to derive the equations for Simple Harmonic Motion beginning with the defining relationship  $a = -n^2 \cdot x$ .

Integrating both sides:

$$\frac{1}{2} v^2 = C - \frac{1}{2} n^2 \cdot x^2$$

2.3 2.4 2.5 2.6 ▸ RAD AUTO REAL

When  $x = a, v = 0 \Rightarrow C = \frac{1}{2} n^2 \cdot a^2$

$$\Rightarrow v^2 = n^2 \cdot (a^2 - x^2)$$

$$\Rightarrow v = |n| \cdot \sqrt{a^2 - x^2} = \frac{dx}{dt}$$

$$\Rightarrow \frac{dt}{dx} = \frac{1}{|n| \cdot \sqrt{a^2 - x^2}}$$

2.4 2.5 2.6 2.7 ▸ RAD AUTO REAL

Integrate both sides wrt x:

$$t = \frac{1}{|n|} \sin^{-1} \left( \frac{x}{a} \right) + C$$

When  $t = 0, x = 0 \Rightarrow C = 0$

$$t = \frac{1}{|n|} \sin^{-1} \left( \frac{x}{a} \right)$$

$\Rightarrow x = a \cdot \sin(n \cdot t)$  as required.

2.5 2.6 2.7 2.8 ▸ RAD AUTO REAL

**Question**

**EX4.** When might it be appropriate to express this equation using cosine instead of sine?

**Answer** ▾