

Introducing the Differential Calculus

The Product Rule

Time required

45 minutes

ID: XXXX

Activity Overview

In this activity, we explore ways to differentiate some more difficult functions. The focus here is on functions which can be expressed as a product of two simpler functions. The approach taken here is largely symbolic and makes full use of the computer algebra facilities of TI-Nspire CAS. Prepared programs and algebraic spreadsheets are also utilized for skill development and consolidation.

Concepts

- Product rule for differentiation, differentiation from first principles.
-

Teacher Preparation

This investigation offers opportunities for review and consolidation of key concepts related to the methods of first principles and differentiation of simple functions. Opportunities are provided for skill development and practise of the method of taking derivatives of products. As such, care should be taken to provide ample time for ALL students to engage actively with the requirements of the task, allowing some who may have missed aspects of earlier work the opportunity to build new and deeper understanding.

- This activity can serve to consolidate earlier work on first principles. It offers a suitable introduction to derivatives of more difficult functions.
- Begin by reviewing the method of differentiation of first principles, both graphically and symbolically, and methods of differentiation of the standard function forms.
- The screenshots on pages X–X (top) demonstrate expected student results. Refer to the screenshots on page X (bottom) for a preview of the student .tns file.
- **To download the .tns file, go to <http://education.ti.com/exchange> and enter "XXXX" in the search box.**

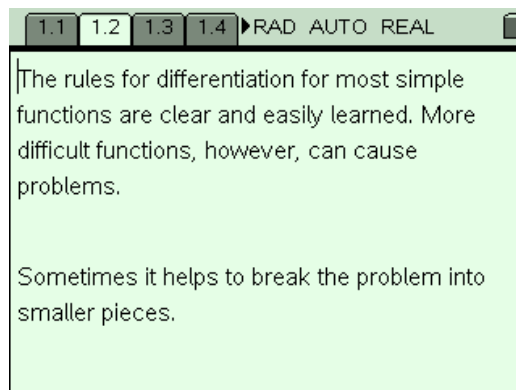
Classroom Management

- This activity is intended to be **teacher led**. You should seat your students in pairs so they can work cooperatively on their handhelds. Use the following pages to present the material to the class and encourage discussion. Students will follow along using their handhelds, although the majority of the ideas and concepts are only presented in **this** document; be sure to cover all the material necessary for students' total comprehension.
- Students can either record their answers on the handheld or you may wish to have the class record their answers on a separate sheet of paper.

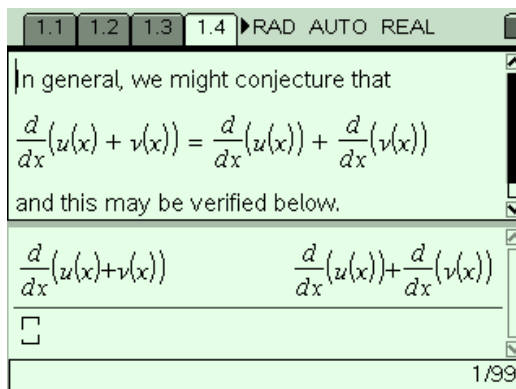
TI-Nspire™ Applications

Calculator, Notes, Lists & Spreadsheet and Programming.

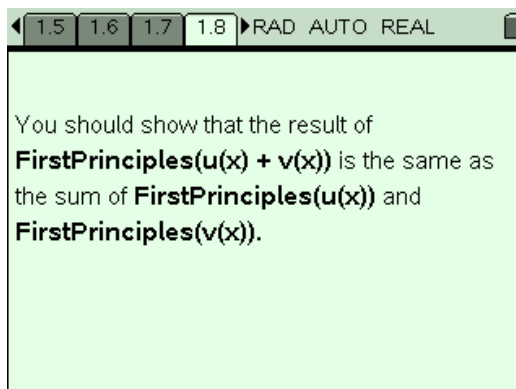
Step 1: Begin with discussion and review of both first principles methods and of the derivatives of standard function forms. Ensure that students are comfortable with these and then challenge them to consider more difficult forms – initially sums, differences and products of functions.



Step 2: After trying some simple examples of sums of functions and their derivatives, students should be asked to conjecture general rules for sums, differences and, finally, products of functions. They may use the CAS (if available) to verify their conjectures as shown.



Step 3: It is one thing to conjecture and verify that conjecture, but quite another to prove such a conjecture. This may be done using the method of First Principles (as shown) and then, if CAS facilities are available, checked using the program **FirstPrinciples(u(x)+v(x)):result**, which shows the step-by-step working and the final result. This offers a valuable guide for students to check their own solutions and a model to help them construct better solutions.



Step 4: Applying these methods to products of functions offers a further challenge: students should discuss ways in which they might check their product conjecture using the given function. Clearly, they can compute derivatives of the expanded form, and of the individual factors and see whether the derivative of a product equals the product of the derivatives (it does not!)

1.10 1.11 1.12 1.13 ▸ RAD AUTO REAL

Question

3. Which brings us to functions made up of a product, rather than a sum or difference.
How might we evaluate $\frac{d}{dx}((x-2) \cdot (x+1))$?

Answer ▾

Step 5: Again, using the **FirstPrinciples(u(x)*v(x))** program, students may be assisted to work towards a verifiable method which reveals the product rule. By typing the variable **result** they have the product rule revealed to them.

1.12 1.13 1.14 1.15 ▸ RAD AUTO REAL

$f(x) = u(x) \cdot v(x)$
Begin with a point on the curve:
 $P(x_1, f(x_1)) = \{x_1, u(x_1) \cdot v(x_1)\}$
Now consider other points close to P
 $Q(x_1+h, f(x_1+h)) = \{h+x_1, u(h+x_1) \cdot v(h+x_1)\}$
 $R(x_1-h, f(x_1-h)) = \{x_1-h, u(x_1-h) \cdot v(x_1-h)\}$
The gradient QR is given by $\Delta y / \Delta x$ where

2/2

Step 6 An **algebraic spreadsheet** is available to support students in working through the process: students supply each step, which is checked for algebraic equivalence. Again, a prepared program may be used to help students in applying and consolidating their learning of this new rule. The **diff_product(fn1, fn2)** program requires students to express a given function as the product of two algebraic factors, and then it displays a step-by-step application of the product rule. Students are asked to simplify the final step, and then check their work by typing **result**.

1.15 1.16 1.17 1.18 ▸ RAD AUTO REAL

Try now with examples such as:

- (1) $2x \cdot (x^2 - 4)$
- (2) $\sin(x) \cdot \cos(x)$
- (3) $x^2 \cdot \ln(x)$
- (4) $(x-2)^2 \cdot (x+1)$
- (5) $2x \cdot \sin(x)$

1.18 1.19 1.20 1.21 ▸ RAD AUTO REAL

	A	B	C
1	u(x) =	2*x	
2	v(x) =	sin(x)	
3	du =		2 √
4	dv =	cos(x)	√
5	d(u*v) =	u*dv+v*du	
B5	"u*dv+v*du"		

Introducing the Differential Calculus: The Product Rule– ID: XXXX

(Student)TI-Nspire File: CalcActXX_Product_Rule_EN.tns

1.1 1.2 1.3 1.4 ▸RAD AUTO REAL

**Introducing the Differential Calculus:
The Product Rule**

Calculus with CAS

1.1 1.2 1.3 1.4 ▸RAD AUTO REAL

The rules for differentiation for most simple functions are clear and easily learned. More difficult functions, however, can cause problems.

Sometimes it helps to break the problem into smaller pieces.

1.1 1.2 1.3 1.4 ▸RAD AUTO REAL

e.g. What about $\frac{d}{dx}(x^2 + x^3)$?

Guess an answer and then check below.

$$\frac{d}{dx}(x^2 + x^3)$$

0/99

1.1 1.2 1.3 1.4 ▸RAD AUTO REAL

In general, we might conjecture that

$$\frac{d}{dx}(u(x) + v(x)) = \frac{d}{dx}(u(x)) + \frac{d}{dx}(v(x))$$

and this may be verified below.

□

0/99

1.2 1.3 1.4 1.5 ▸RAD AUTO REAL

Question

1. How might you prove that the derivative of a sum is the sum of the derivatives?

Answer ▾

1.3 1.4 1.5 1.6 ▸RAD AUTO REAL

Question

2. Does the same hold for differences?

Answer ▾

1.4 1.5 1.6 1.7 ▸RAD AUTO REAL

These results may be shown to always hold true using the method of differentiation by First Principles. You should review this method and try to apply it here.

To check your proof, you might run the program, **FirstPrinciples(u(x) + v(x))** then type the variable, **result**, to see the final result.

1.5 1.6 1.7 1.8 ▸RAD AUTO REAL

You should show that the result of **FirstPrinciples(u(x) + v(x))** is the same as the sum of **FirstPrinciples(u(x))** and **FirstPrinciples(v(x))**.

1.6 1.7 1.8 1.9 ▸RAD AUTO REAL

firstprinciples(u(x)+v(x)).result

$$f(x) = u(x) + v(x)$$

Begin with a point on the curve:

$$P(x_1, f(x_1)) = \{x_1, u(x_1) + v(x_1)\}$$

Now consider other points close to P

$$Q(x_1 + h, f(x_1 + h)) = \{x_1 + h, u(x_1 + h) + v(x_1 + h)\}$$

$$R(x_1 - h, f(x_1 - h)) = \{x_1 - h, u(x_1 - h) + v(x_1 - h)\}$$

1/3

1.7 1.8 1.9 1.10 ▸RAD AUTO REAL

Question

3. Which brings us to functions made up of a product, rather than a sum or difference.

How might we evaluate $\frac{d}{dx}((x - 2) \cdot (x + 1))$?

Answer ▾

1.8 1.9 1.10 1.11 ▸RAD AUTO REAL

Question

4. Find the derivatives of $(x - 2)$ and $(x + 1)$ and multiply these together.

Answer ▾

1.9 1.10 1.11 1.12 ▸RAD AUTO REAL

Question

5. Now expand $(x - 2) \cdot (x + 1)$ and find the derivative of the parts.

Answer ▾

1.10 1.11 1.12 1.13 ▶ RAD AUTO REAL

It appears that the derivative of a product is NOT equal to the product of the derivatives. How might we find a rule for the derivative of a product function of the form $u(x) \cdot v(x)$?

1.13 1.14 1.15 1.16 ▶ RAD AUTO REAL

Question

6. Once again, try using First Principles to evaluate this result, and check your answer using the **FirstPrinciples(u(x)*v(x))** program.

Answer

1.13 1.14 1.15 1.16 ▶ RAD AUTO REAL

In general, $f'(x) =$

$$\lim_{h \rightarrow 0} \left(\frac{-u(x-h) \cdot v(x-h) - u(x+h) \cdot v(x+h)}{2 \cdot h} \right)$$

Done

result $\frac{d}{dx}(u(x)) \cdot v(x) + \frac{d}{dx}(v(x)) \cdot u(x)$

2/99

1.13 1.14 1.15 1.16 ▶ RAD AUTO REAL

While the program gave us the result we desired, it did NOT actually provide all the steps needed to understand that result. The last line of the working gives $f'(x) =$

$$\lim_{h \rightarrow 0} \left(\frac{-u(x-h) \cdot v(x-h) - u(x+h) \cdot v(x+h)}{2 \cdot h} \right)$$

But how do we get to the final result?

1.14 1.15 1.16 1.17 ▶ RAD AUTO REAL

One method for simplifying this limit is to add zero to the numerator – but a special version of zero: Clearly,

$$u(x-h) \cdot v(x+h) - u(x-h) \cdot v(x+h) = 0$$

Adding this to the numerator gives a result which can be factored.

$$\frac{u(x+h) \cdot v(x+h) + [u(x-h) \cdot v(x+h) - u(x-h) \cdot v(x-h)] - u(x-h) \cdot v(x-h)}{2 \cdot h}$$

1.15 1.16 1.17 1.18 ▶ RAD AUTO REAL

Question

7. Break this expression into two halves and factor each. Then take the limit as h approaches 0 and give the result.

Answer

One half =

1.16 1.17 1.18 1.19 ▶ RAD AUTO REAL

Try now with examples such as:

- $2x \cdot (x^2 - 4)$
- $\sin(x) \cdot \cos(x)$
- $x^2 \cdot \ln(x)$
- $(x-2)^2 \cdot (x+1)$
- $2x \cdot \sin(x)$

1.17 1.18 1.19 1.20 ▶ RAD AUTO REAL

Use the spreadsheet on the next page to work through these exercises. Enter the two factors of your product function into cells B1 and B2 ("u" and "v"). Enter the derivatives of these functions into cells B3 and B4 ("du" and "dv"). Finally enter the product rule into B6. Each of your steps will be checked.

1.18 1.19 1.20 1.21 ▶ RAD AUTO REAL

	A	B	C
1	$u(x) =$?	
2	$v(x) =$?	
3	$du =$?	X
4	$dv =$?	X
5	$d(u \cdot v) =$	$u \cdot dv + v \cdot du$	
B1		$u := "$	$?"$

1.19 1.20 1.21 1.22 ▶ RAD AUTO REAL

On the next page, you may check your results using the program **diff_product(u(x), v(x))** which gives the product rule differentiation for the function, $u(x) \cdot v(x)$. To see the final result of this program, type the word "result".

1.20 1.21 1.22 1.23 ▶ RAD AUTO REAL

$d(v) = \cos(x)$

$$d(u \cdot v) = (2 \cdot x) \cdot (\cos(x)) + (\sin(x)) \cdot (2)$$

Try to evaluate this then check your answer by typing <result>.

Done

result $2 \cdot x \cdot \cos(x) + 2 \cdot \sin(x)$

10/99

1.21 1.22 1.23 1.24 ▶ RAD AUTO REAL

We have established that

$$\frac{d}{dx}(u(x) \cdot v(x)) = \frac{d}{dx}(u(x)) \cdot v(x) + \frac{d}{dx}(v(x)) \cdot u(x)$$

More simply, this result is often given as

$$\frac{d}{dx}(u \cdot v) = u \cdot dv + v \cdot du$$

(The first times the d/dx of the second plus the second times the d/dx of the first,