Learning to Use New Tools:
A study of mathematical software use for the learning of algebra

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A thesis submitted in fulfilment

of the requirements for the degree of

Doctor of Philosophy

School of Teacher Education

University of New South Wales

August 1995

DECLARATION

I, Stephen Mark Arnold, the undersigned, hereby declare that this submission is my own work and that, to the best of my knowledge and belief, it contains no material previously published or written by another person nor material which to a substantial extent has been accepted for the award of any other degree or diploma of a university or other institute of higher learning, except where due acknowledgment is made in the text.

SIGNED: ....

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ABSTRACT

This study documents the efforts of a teacher/researcher to learn to use computer algebra software applications as pedagogical tools through systematic self-study, clinical observations of secondary students and collaboration with groups of preservice teachers. The study also involved the ongoing development of a computer-based learning environment which accompanied the research process and served to embody the main results. Complementing action research methods with grounded theory analysis, the study describes and explains the ways in which individuals (six secondary students and two groups of six preservice teachers) used available software tools for algebra learning. The subsequent grounded theory situates tool use within contexts of mathematical and pedagogical thinking on the part of the user. Effective use of available software tools was also found to be conditional upon characteristics of both the software and the learning environment.

Analysis of pedagogical beliefs of both students and preservice teachers revealed a consistent culture of mathematics learning which devalued external support factors and exploration in favour of repetitive individual skill development within teacher-dominated instructional sequences. Detailed analysis of students' algebraic imagery revealed that, while some algebraic forms served a strong and consistent signal function in eliciting meaning and action strategies, others (including simple expressions and tables of values) were associated with unclear signals, frustrating students' abilities to act appropriately in both traditional and computer-based learning situations. These factors acted as impediments to the effective use of mathematical software tools.

At the same time, *strategic* use of appropriate mathematical software (defined as goal-directed, flexible and insightful) supported the development of algebraic skills and understandings in students. Such use was associated for the students with increased manipulative and representational repertoires and increased confidence in their results. The graphical representation was most favoured by all participants, although it was commonly associated with superficial and automatic use. The table of values, while recognised as effective for detailed comparison of functions, was more difficult to interpret and less favoured. Computer algebra tools were found to be most effective in supporting mathematical investigation and the explicit development of extended algebraic processes, such as equation solving.

ACKNOWLEDGMENTS

This study was made possible through support received from the School of Teacher Education at the University of New South Wales in the form of an Australian Post-graduate Research Award, from the Australian Association of Mathematics Teachers, from EDSOFT Australia and The SoftWarehouse (for donating copies of *Derive*), and from Thomas Nelson Australia and Prescience (for donating copies of *Theorist Student Edition*). My thanks go to my supervisors, Dr John Scheding and Professor Michael Dunkin for their patient and wise advice.

Special thanks must go to my wife, Kerry and our children, Anthony, Patrick, Naomi and Nathan, who shared so much of the experience and made so many sacrifices to allow me to pursue my dream.

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BiPlane 2.0 Shareware Spreadsheet. Published by Night Diamonds Software.

Calculus T/L II by J. Douglas Childs. Published in Australia by Thomas Nelson Australia.

CC the Calculus Calculator by David Meredith, San Francisco State University.

CoCoA (Commutative Computer Algebra) by Allessandro Giovini and Gianfranco Niesi, Department of Mathematics, University of Genova, Italy.

DERIVE v2.5. Published by the Soft Warehouse, Hawaii, and distributed in Australia through EDSOFT (P.O. Box 314, Blackburn VIC. 3130).

Mathematica Student. Published by Wolfram Research.

Maple V Student Version. Published by Brooks Cole Publishing.

MathMaster 2.21 by Paul Cozza.

Milo™ 1.00 by Ron Avitzur.

Theorist Student Edition. Published by Waterloo Maple.

Xplore - the Mathematical ToolChest by David Meredith. Published by Prentice Hall Australia

One *Introduction*

> *"If one changes the tools of thinking available to a child, his mind will have a radically different structure."* (Vygotsky, 1978, p. 126)

Mathematics learning is a complex activity. The thinking which accompanies and directs such learning contains both mathematical and pedagogical elements, as those involved engage in process and content dimensions of the activity, within a variety of learning contexts. At different times, mathematics learning may be seen to incorporate elements indicative of distinct cognitive modes - from sensori-motor (utilising physical movement and goal-directed action) and ikonic (global, intuitive and visual) to the concrete-symbolic and formal modes, more usually associated with mathematical activity across the secondary school and beyond. Of critical importance, however, is a conception of mathematics as a "tool-based activity" (Confrey, 1993a), supported and made possible by such cognitive aids as language (both informal and formal), notation and symbol systems. Additionally, there are external tools for mathematics learning - writing and drawing implements, calculators, geometric construction instruments and, increasingly, computer hardware and software. These last appear to offer new means of transforming, not only the teaching and learning process, but perhaps the nature of mathematics itself (Steen, 1992, Kaput, 1992, Bishop, 1993).

This study of the use of advanced mathematical software by secondary students and preservice teachers attempts to make explicit the ways in which individuals think and act when doing and learning mathematics in a tool-based context. The action occurs within the domain of algebra, long a focal point for mathematics education research. The importance accorded to the learning of algebra is a consistent feature of educational systems worldwide (Wagner and Kieran, 1989, Kieran, 1992); equally consistent is its place as a principal stumbling block for learners. The impact of computer technology upon the twin processes of teaching and learning within the domain of algebra adds a new, potentially explanatory dimension to this problematic field of study.

The research focus lies with the use of mathematical software tools within the context of algebra learning situations. Tools which support manipulative algebra and multiple representations of mathematical objects and processes are now well recognised as means of enhancing mathematical activity (particularly at post-school levels), but they have been little explored as tools for pedagogy and instruction. This study documents the path followed by one teacher (the researcher) as he systematically studies the use of advanced mathematical software applications as tools for both pedagogy and exploration. As a practitioner seeking to improve his own practice through systematic, reflective and (at times) collaborative activity, the author builds upon the research foundation offered by the action research tradition (Lewin, 1946, Carr and Kemmis, 1983, Kemmis and McTaggart, 1988a, 1988b) and blends this approach with the methodological and analytical rigour offered by the Grounded Theory method (Glaser and Strauss, 1967, Strauss and Corbin, 1990).

It is the purpose of this chapter to introduce the principal concerns and direction of this study. Beginning with a consideration of the nature of the research design as a blending of action research and grounded theory, the focus moves from an overview of the role and nature of the new tools now available for mathematics teaching and learning, to an attempt to place the learning of algebra in the secondary school within the context of international, national and state developments since 1980. Attention is then turned to the theoretical bases from which the study initially springs: the theories of learning which underpin the approach taken and the research principles which guided the design. Finally, the unique role of the computer as both focus of study and primary tool for data collection is considered.

As an action research project, the study begins with the identification of a problem. In the present context, this problem revolves around the desire of the researcher as teacher and tutor to "learn to use new tools" - to acquire skills and knowledge of ways in which algebra software tools might best be incorporated into individual learning situations, as tools for pedagogy as well as mathematics. *Doing mathematics* may be considered in terms of an interplay between *grounded activity* and *systematic enquiry* (Confrey, 1993a, pp. 51-54). Mathematical software tools further emphasise this dialectic, as tools naturally linked with action, and yet supporting and encouraging open-ended exploration of mathematical ideas. Mathematics learning may also be situated at various points along a continuum, created by the tension between *instruction* (characterised by teacher-centred activities, in which knowledge is transmitted to the students, who play a largely passive role in the process) and *enquiry* (in which students create meaning from the learning situation as active participants responsible for their own learning). The role of mathematical software tools within this framework is problematic. It is the purpose of this study to explore the mathematical and pedagogical dimensions of both thinking and action within algebra learning situations, and to make explicit the role of technology within the process.

The project follows a series of distinct action cycles through which the research design is played out and the features and nature of the problem made explicit (Figures 1.1 and 1.2). The various cycles within the design may be recognised as occupying distinct dimensional spaces, aligned with four critical actions which define the study:

(1) the reflections and activities of the researcher himself, which drive and direct the study;

(2) observations of individual school students learning to use the available technology as both *mathematical* and *pedagogical* tools within technology-rich algebra tutorial situations;

(3) collaboration with two groups of preservice teachers, one group focusing particularly upon pedagogical use and the other upon mathematical use of the software tools; and

(4) the continuing redefinition of the notion of a *technology-rich learning environment*, which provides the context for the study. Such an environment is seen to be defined in terms of three critical variables: pedagogical *content* (the focus of the learning experience), pedagogical *action* (the actions of both teacher and students by which learning is enabled), and the

nature of the tools themselves, which are open to development. In this way, the software tools themselves become an embodiment of the reflections and interpretations of the research process by the researcher. Each of these critical variables provides a basis for revision of the learning environment through the various action cycles of the study.

The overall research design of this study clearly reflects the influence of action research methods, as a practitioner seeking to improve his own practice systematically investigates his own use of algebra software tools, and that of others, through a series of action spirals. Each action cycle begins with the researcher **planning** and **acting** to create a particular interactional situation - what is termed here a *technology-rich algebra learning environment*. **Observing** and **reflecting** upon these observations gives rise to a **revised plan**, embodied in revisions to the curricular content, the pedagogical actions or in physical changes to the available software tools in response to the previous cycle.

Figure 1.1: The Action Research Spiral

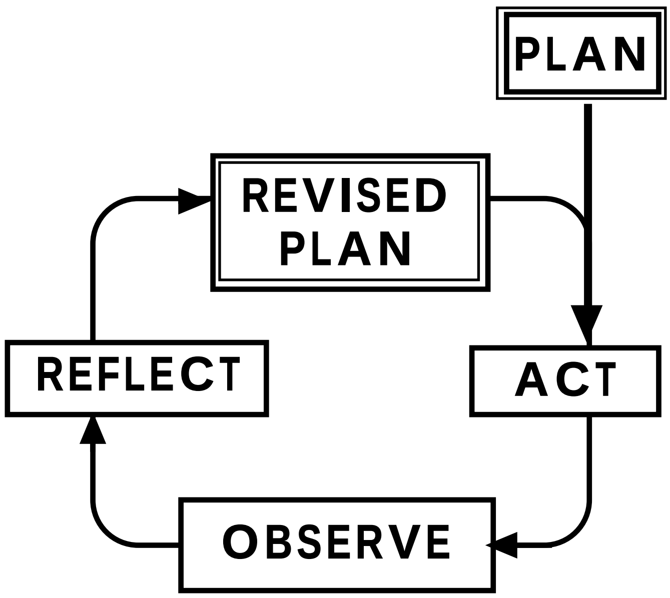
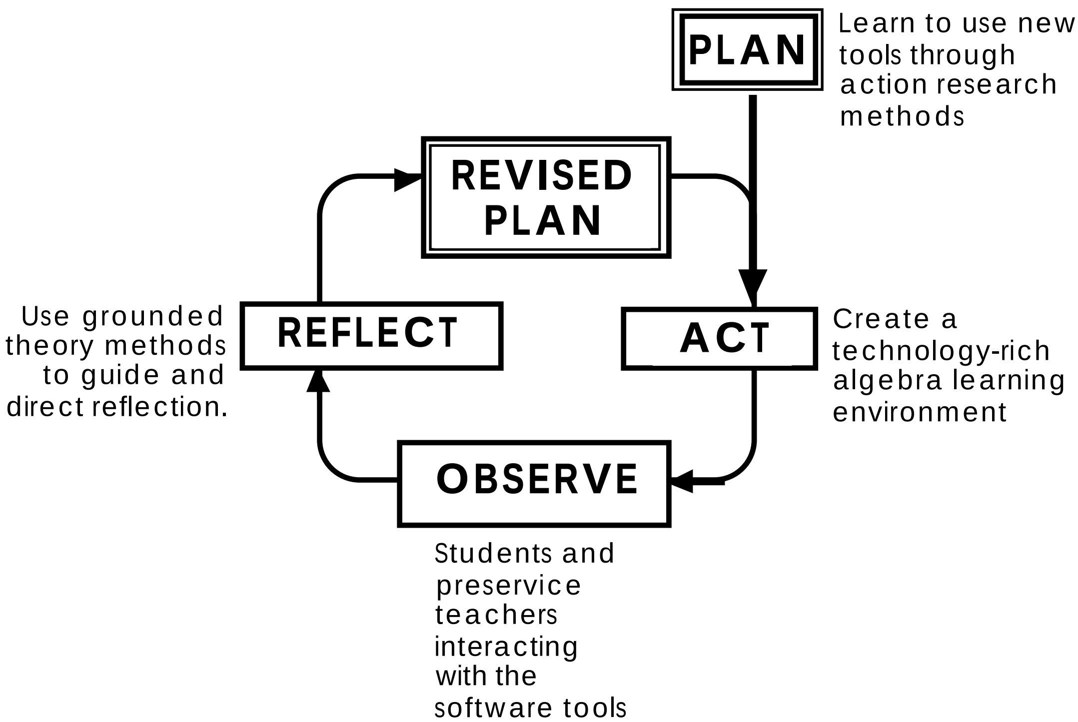


Figure 1.2: The action research design



Miles and Huberman argue convincingly that the weakest link in the chain of qualitative research is that between data and conclusions

(Miles and Huberman, 1984, Huberman and Miles, 1994). Although action research provides a powerful methodological foundation for the study, it offers little to guide the data analysis process. The *grounded theory method* (Glaser and Strauss, 1967, Strauss and Corbin, 1990) has been adopted for this study, offering a detailed and systematic approach to the qualitative research process which provides an ideal complement to the action research methods described above. The use of a grounded theory approach to the analysis of data supports and encourages the development of a theoretical framework which is grounded in the data but potentially applicable beyond the immediate situation. Such a theory is intended to speak directly to practitioners, and to offer guidance and direction to both future research and practice. The merging of action research with grounded theory offers a powerful research design for the current study, clearly defining the reflective act so central to action research.

The focus of a grounded theory design is upon a central phenomenon (or *core category*), a slightly different perspective to the *significant problem* which gives rise to an action research study. By identifying *tool use within algebra learning situations* as the core category, however, the critical link between the two perspectives may be established. The central concern of the researcher in the study which follows, then, is *learning to use new tools*, initially, on a personal level and, subsequently, involving the students and preservice teachers who are drawn to interact with the available software tools within various algebra learning contexts. All participants share this common goal, although each approaches it from a distinct perspective. The unifying phenomenon, or core category, for all participants is that of *tool use*.

In particular, this study explores a phenomenon described here as *strategic software use* - the deliberate, flexible and goal-directed use of available tools as a means of increasing understanding and offering insight into problematic features of the particular learning context. Such use may be seen to represent an ideal, a high level of attainment with regard to both the mathematical learning context and the cognitive tools used within this context - it may be recognised as lying at one end of a continuum along which may be found various styles and dimensions of tool use. The defining characteristics of this style of use, the context within which it occurs and develops, the forms it takes within the interactional process involving user, tool and object of study, and the short- and long-term consequences - all are critical aspects of strategic software use which this study seeks to explicate.

It is hypothesised that thinking and tool use exist within a recursive relationship - each influencing and significantly altering the nature of the other. While this influence is likely to be most evident within a framework of *strategic* use, it is recognised at the outset that such use is likely to be rare. Factors which both encourage and inhibit such strategic use must be made explicit, as must the nature of the use itself, and its consequences.

The research takes as its principal focus the use of a generic software type: symbolic manipulation or, more usually, computer algebra software, by students and student teachers. The use of the software occurs within the context of an algebraic learning environment created for this purpose to provide access and direction in the use of these and other advanced software tools (particularly graph plotting and table of values utilities). The study seeks to document and explain the ways in which individuals use the various tools available, the context of this use, and aspects of the proposed relationship between tool use and their mathematical and pedagogical thinking. As such, it is interpretative in nature and qualitative in design.

The design of the study and the subsequent gathering of data are driven by four principal research questions. The first two of these define *algebraic thinking* and the third *pedagogical thinking* as the terms are used for the purposes of this project. The fourth defines *tool use* as it is to be considered in the present context.

- What do individuals (researcher, students and preservice teachers) understand by algebra and its components (especially functions, variables, equations, graphs and tables of values) and how might such understandings be related to the use of computer tools?

- What do individuals perceive when they view algebraic objects and how may these perceptions influence their choice and use of available strategies (including the use of mathematical software)?

- What beliefs do individuals bring with them to algebra learning situations concerning the nature of algebra, the ways in which it may best be learned, and the characteristics of successful learning and effective teaching practice? To what extent may such beliefs impact upon the use of technology as a learning strategy?

- Under what conditions do individuals choose to use available software tools, and what forms does this use take? What features of both tool and learning situation serve either to impede or encourage such use?

These themes of algebraic thinking, pedagogical thinking and tool use dictate the concerns and direction of the study which follows. They also preface the structure of the chapters of this book.

The thesis falls into two main sections, Chapters One to Four laying the foundations in terms of the literature, tools and research design, and Chapter Five serving as a bridge, setting out the principal categories by which the data was initially conceptualised and providing insight into both the researcher's approach and beliefs and the subsequent development of a grounded theory of mathematical software use. Chapters Six, Seven and Eight set out in detail the responses of the participants concerning issues associated with algebraic and pedagogical thinking and tool use, as outlined above. Finally, these results are drawn together using the Grounded Theory paradigm in Chapter Nine, leading to the development of a rich and interconnected theory of mathematical software use for the learning of algebra.

New Tools for Mathematics Learning

During the 1970s, the impact of early computer technology made itself felt upon mathematics instruction in the form of hand-held calculators. Soon recognised as offering enormous potential for change in the teaching and learning of mathematics at all levels, there existed at the same time a reluctance on the part of many teachers to freely embrace such tools. Unsure of how to change their teaching so as to effectively incorporate the new technology, they experienced an initial hesitation concerning the effects and possible dangers. Twenty years later, through extensive research and classroom use, many of the early misgivings have been laid to rest, many of the pitfalls recognised (Hembree and Dessart, 1992). The scientific calculator is now assured of its place as an essential adjunct to the learning of mathematics.

This situation seems destined to repeat itself in the 1990s, not with numerical tools but with algebraic ones. Once more, teachers of mathematics are presented with a dilemma - the availability of technologies capable of enhancing the teaching and learning of their subject, but a lack of insight and clear direction as to how they may most profit from their use. Once more, “(t)he first instinct of educators is to couple the new technology to their old methods of instruction” (Papert, 1980, p. 230). While significant work has centred upon the potential of computer tools such as spreadsheets and graph plotters (particularly through the ongoing Technology-Enriched Algebra Project (Asp, 1991, Asp, Dowsey and Stacey, 1992, 1993a, 1993b) and recent research by Quinlan (1994)) the potential role of manipulative algebra software within secondary school settings remains largely unexplored.

Since 1980, computer software capable of performing all of the algebraic manipulations required for high school and beyond has been available for microcomputers. Initially designed for use by engineers, scientists and research mathematicians, such tools, though powerful, were difficult to learn, and largely unsuited to the secondary classroom. Early “computer algebra systems” have given way to a more sophisticated and appropriate tool in the last five years, capable of the representation and manipulation of mathematical functions in at least algebraic and graphical forms. Further, recent software has been designed to present these mathematical objects using correct mathematical notation, thus removing the need for students to learn a "computer algebra syntax" in addition to the algebraic syntax already required of them. By allowing the user to select items from menus, templates or palettes, and so to construct and manipulate mathematical expressions with relative ease, such tools at last appear appropriate to incorporate into high school learning situations.

"Enhanced computer algebra software" in its various forms allows the user to represent functions algebraically, graphically and numerically; ideally the user may move interactively between these representations. Additionally, some manipulation of these forms is possible. The algebraic representation, for example, may be rearranged, expanded, simplified and, in some cases, factorised. It may be differentiated and integrated, numerically and often symbolically. Exact arithmetic calculations are often possible, allowing work with fractions, surds and complex numbers in exact forms, or to any desired degree of accuracy. The graphical form may be rescaled by adjusting the "viewing window"; "zooming" in or out to allow the function to be observed in a wider context, or in finer detail. Points of interest may be isolated and identified. Equations may be solved graphically or algebraically. The power of computer algebra is linked to the versatility of the graph plotter. The computer becomes a means of exploring the nature and properties of the functions which make up the larger part of the study of mathematics at the higher levels, and of relieving much of the syntactical burden which proves a barrier for many learning algebra.

The availability of such tools presents enormous implications for change in the ways in which mathematics may be learned. Much of the time spent in high school mathematics courses currently arises from the perceived need for students to acquire relative mastery of the skills of algebraic manipulation. The increasing availability of computer tools which will perform these manipulations more quickly and accurately than was previously possible suggests that such time may be spent more profitably in other ways. In particular, reducing the time spent on the mastery of manipulative skills may allow more time to be spent on activities which will aid in concept development and understanding, and on the applications of mathematics to “real-world” situations (Heid, 1988, 1989). Such features appear sadly lacking from most senior mathematics courses, both here and overseas (Tobin and Fraser, 1988, pp. 77-79).

The use of such computer technology permits three fundamental advantages which would otherwise be too difficult, or quite impossible to achieve within more traditional mathematics learning situations:

- *New Modes of Representation:* The use of computer tools facilitates a variety of representations which may serve to deepen understanding and encourage investigation.
- *Power of Calculation:* Computer tools are capable of performing both numerical and symbolic manipulations quickly and accurately, freeing students to focus upon the context and meaning of the problem situation, and to investigate applications which might otherwise be too difficult or inaccessible to them.
- *Interactivity:* Computers are dynamic tools, quite distinct from the passive aids which characterise many school experiences.

> The ease with which the student may direct and control the computer, accessing powerful mathematical features further justifies their inclusion in the curriculum.

The parallels with the advent of hand-held calculators are precise. Activities devoted to particular skill development, such as the long division algorithm, were found to require a disproportionate amount of class time. The decreased emphasis upon such skills which has resulted from the widespread use and availability of calculators has freed such time for other purposes. It seems likely that tools capable of algebraic manipulation and the multiple representation of functions may also become a means of freeing up the curriculum for activities determined to be more appropriate than the repetitive skill development which is currently the norm.

Finally, such tools call into question existing assumptions regarding the nature of teaching and learning in mathematics. Instructional practices which have been considered successful and, in many cases, “expert”, are being challenged by this technology. All too often, such practices are being exposed as encouraging rote learning and superficial understandings (see, for example, Schoenfeld, 1988). Those who have relied upon these practices may well feel threatened by recent calls to incorporate the new technology as one means of increasing the emphasis upon understanding and conceptual development in mathematics learning.

This study potentially offers new knowledge of the ways in which those learning algebra engage in the use of available software tools. Such considerations are critical in seeking to understand ways in which such tools may be effectively used as means to increase understanding and enhance algebra instruction at all levels.

Algebraic Thinking in Secondary Schools

The teaching and learning of algebra in the secondary school have become objects of increasing attention worldwide over the past decade. The beginning of the 1980s appeared to be a time of great resolve and desire for change in the international mathematics education community, typified by the large-scale *Concepts in Secondary Mathematics and Science Project* (C.S.M.S., 1980-82) and the *Cockcroft Report* (1982) in Great Britain, and the National Council of Teachers of Mathematics (N.C.T.M.) *Agenda for Action* (1980) in the United States. More recently, the release of the N.C.T.M. *Curriculum and Evaluation Standards* (N.C.T.M., 1989) and, here in Australia, the Australian Education Council's documents, *A National Statement on Mathematics for Australian Schools* (Australian Education Council, 1990) and the subsequent *National Mathematics Profiles* (Curriculum Corporation, 1993) suggest that the momentum has been sustained. Australian work on "concrete approaches to algebra" (Booth, 1989, Quinlan *et al*, 1989 and Pegg and Redden, 1990) appear to be indicative of recent research efforts. Certainly, with their inclusion in recently released Syllabus documents in New South Wales and Queensland, and consequently their inclusion in the new generation of textbooks for junior mathematics, there are hopeful signs that teachers may begin to move away from the traditional "rote learning" approach to the teaching of algebra.

The *National Statement on Mathematics for Australian Schools* (Australian Education Council, 1990, pp. 187-189) describes algebra at all levels of schooling as serving three fundamental purposes:

- *Expressing generality* : As the primary means by which we may describe and understand patterns and generalisations in a multitude of situations.

- *Describing Functions* : Providing the means by which functional relationships may be defined and represented using a variety of forms, especially numerical, graphical and symbolic.

- *Solving Equations* : These form the essential mathematical problem solving tools by which problem situations may be analysed and described mathematically, and then redefined in a simpler or more accessible form.

Distinguishing between the “syntax” of algebra, in terms of the rules and manipulations which govern its use, and the “semantics” (the meaning and context of algebraic activities), school mathematics has been dominated by the former. A critical pedagogical role of computer algebra, then, may lie in its ability to perform the syntactical operations of algebra, and so to allow greater emphasis upon context, meaning and applications. Used appropriately, it should also permit students increased opportunities for reflection upon the processes involved, rather than being absorbed by the manipulations themselves.

While there are grounds for optimism that algebra learning in the junior school may be expected to become more concrete and, hopefully, more meaningful for students, there appears to be little of the same pressure at the higher levels. Middle school mathematics courses (Years 9 and 10) are still dominated by skills of manipulation, largely in preparation for the requirements of the calculus in the senior school. Students attempting the calculus-based courses in Years 11 and 12 are expected to be proficient in algebraic manipulation, possibly to the exclusion of other skills. Students without solid grounding in this area may not be expected to succeed, even at the “average” senior level (which, in New South Wales, is the 2 Unit Mathematics course).

With retention rates into senior education in Australian schools reaching 80% and with ever-increasing requirements for tertiary entrance and employment, schools are feeling growing pressures to admit students to such courses who would not previously have been considered. Enrolment patterns in senior mathematics courses offered in New South Wales over the period from 1983-93 may be considered representative of those in other Australian states, and are depicted in Figures 1.3, 1.4 and 1.5 (see Appendix A for the corresponding numerical information). The graphs illustrate the percentages of the total student candidature for that year attempting the four courses available (Figure 1.3), the actual numbers of students for each course (Figure 1.4) and, finally, the percentage increase for each course over the eleven year period.

Figure 1.3: NSW HSC Mathematics Courses 1983-93: Percentage of Total Candidature

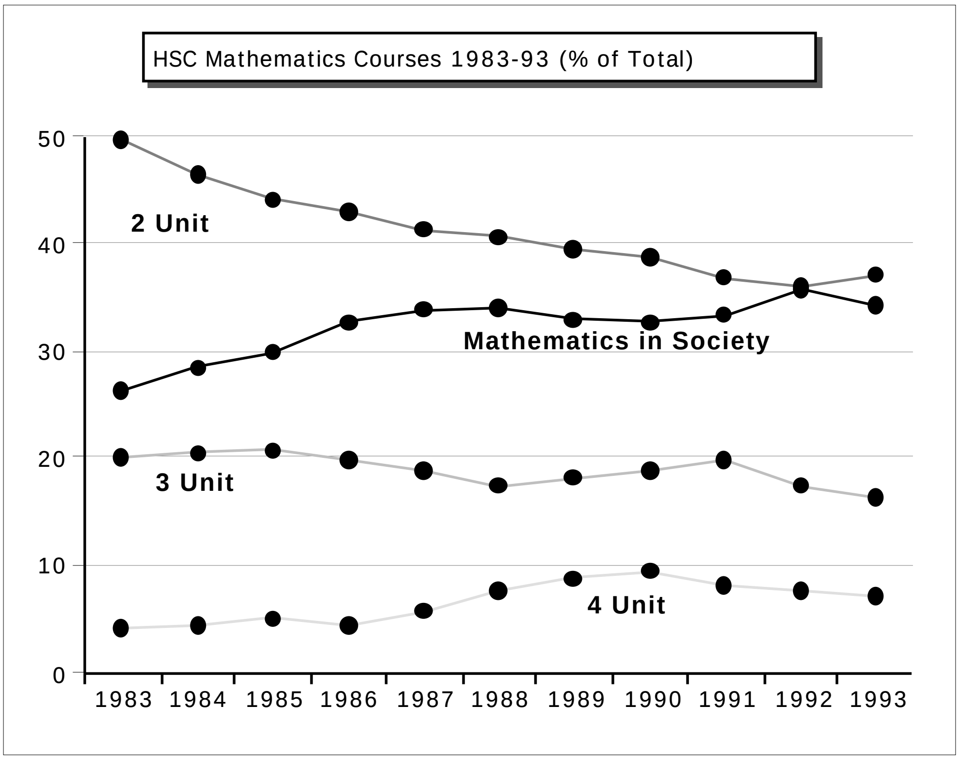


Figure 1.4: NSW HSC Mathematics 1983-93: Numbers of Candidates

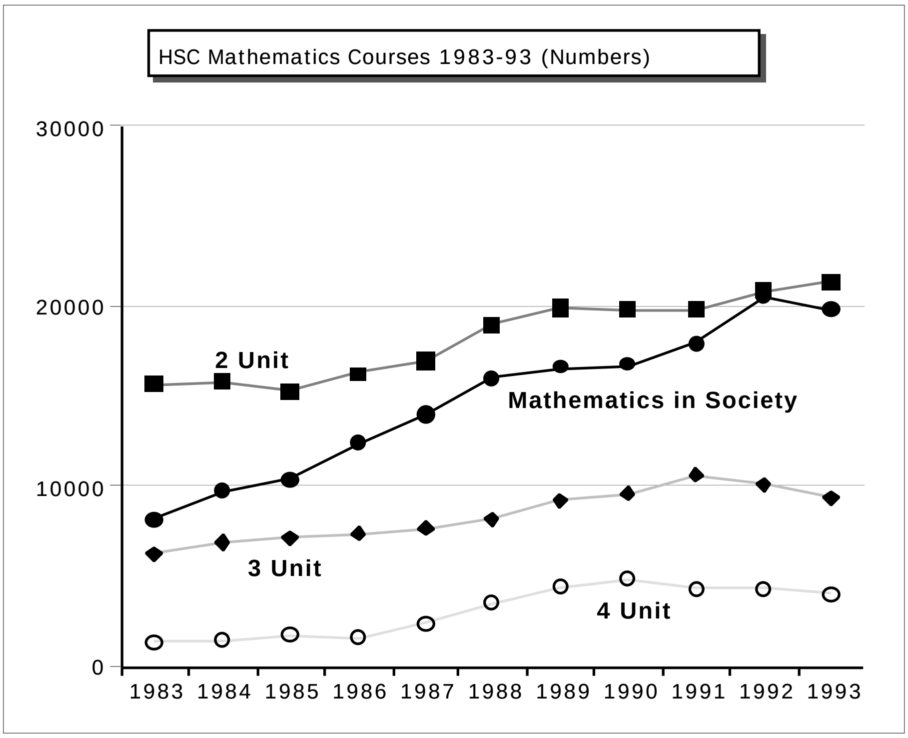
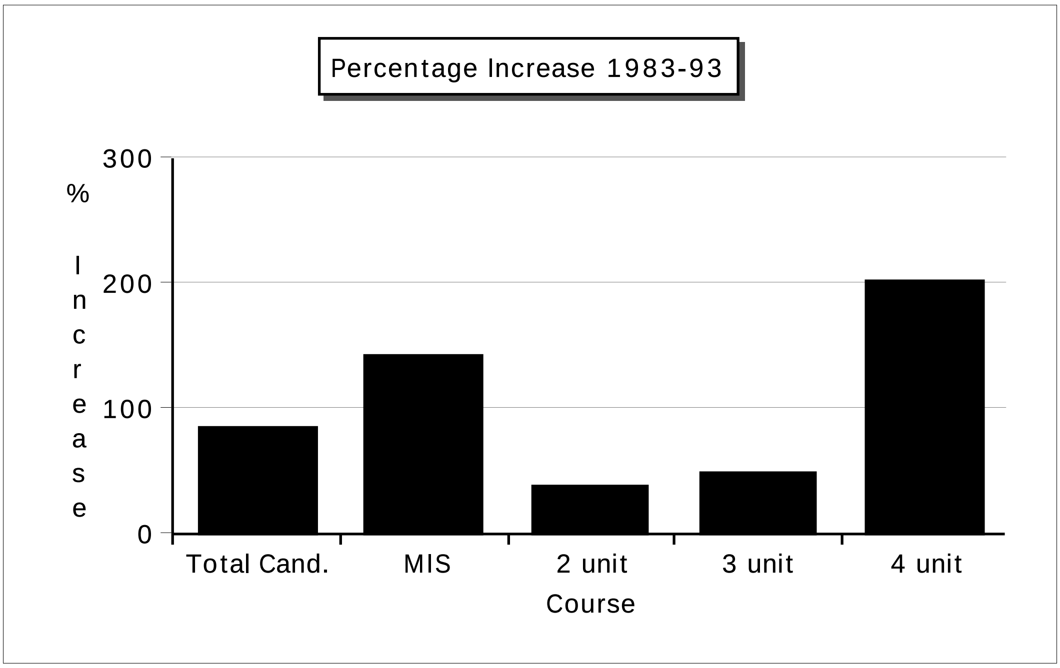


Figure 1.5: NSW HSC Mathematics Courses 1983-93: Percentage Increase



Of the four courses offered, the *Mathematics in Society* course is a non-calculus option, considered as a terminating course for purposes of tertiary mathematics study. (A fifth course was offered for the first time in 1991, intended for students at a lower level than that catered for by the *Mathematics in Society* course. This course, *Mathematics in Practice*, accounted for only about 5% of the candidature in 1993.) Those courses labelled *2 unit, 3 unit* and *4 unit* are provided as tertiary preparation courses for mathematics, and are characterised by increasing breadth and depth of content. Of critical significance with regard to interpretation of these graphs is the large increase in the total candidature in mathematics for the Higher School Certificate over this period, from 31,448 students in 1983 to 57,709 in 1993. This is indicative of the increasing enrolments across all subject areas in the senior years over this period.

Within the context of this increase in overall numbers, the graphs reveal significant changes in enrolment patterns. The greatest increase over this period has occurred in the *Mathematics in Society* course. Although the *2 Unit* course still accounts for the highest overall proportion, it has dropped significantly in this regard. Clearly, the majority of the "new" students (those who would probably not have continued on to senior study in 1983) are enrolling in the less demanding course. At the same time, the actual numbers of students in all courses have increased - there are physically more students attempting all mathematics courses than there were a decade ago. Interestingly, the greatest increase overall (when balanced against the increased student population) has occurred in the highest course, the *4 Unit* course. It seems likely that syllabus changes to this course in 1986 served to make it more accessible than previously; it has also benefited from a significant scaling advantage when scores for tertiary entrance are computed.

Within the secondary school, algebra has traditionally served as a "gatekeeper" for study at the higher levels, performing in many instances a deliberately discriminatory role in separating students. Just as a lack of proficiency in numerical skills may serve to deny many students opportunities to engage in mathematical content areas available to their peers, so too does poor algebraic facility. As hand-held calculators have served a scaffolding role, supporting students in their study of these aspects of mathematics previously denied them, so too may algebra software expand the options for many, not just in the senior years, but across the secondary school and perhaps beyond. Questions as to the effectiveness and potential of advanced mathematical software tools in the learning of algebra, then, become both educationally and socially relevant.

Theoretical Frameworks

In seeking to describe and understand the ways in which students and teachers think about and use advanced mathematical software, and the consequent effects of such use, the role of theory is perceived to be one of providing initial guidance and direction in a fledgling field of enquiry. In the model developed for this study, the computer is perceived as being, at the same time, both focus and method of enquiry. Advanced mathematical software, as it is used by individuals involved in mathematics learning, provides the principal target for description and explanation. As the primary mode of data collection in the research design, the computer, too, becomes the central means of enquiry, as it provides an ideal tool by which such interactions may be made explicit.

No single theoretical model appears sufficient to provide description and explanation at a suitably rich and meaningful level. Rather, a melding of several compatible and complementary theories of learning and cognition allows the complex interactions of individual and technology to be categorised in ways which illuminate different aspects of the process. Thus, at what might be considered the “micro” level, the SOLO taxonomy (Biggs and Collis, 1982, 1991, Collis and Biggs, 1991, Collis, Watson and Campbell, 1992) provides a detailed descriptor, not only of the various modes of thinking which characterise the cognitive activities of different individuals at different times, but also of the developmental sequence occurring *within* each of these modes. The taxonomy fails, however, to provide adequate explanation for the ways in which learners move between the various modes of thinking, and the potential role of teacher (and technology) in assisting such transition.

A theory of learning proposed by Pierre van Hiele (1986), specific to mathematics education, offers a different perspective on the modes of thinking, but one which is consistent with that of the SOLO taxonomy. There is, in this way, the possibility for an increased richness of description as each theory illuminates the same aspects from a different angle, each casting light upon features which the other may fail to fully accentuate. Further, van Hiele is concerned primarily with the role of the teaching process in the transition between levels of thinking, and offers much in the search to explain the ways in which teacher and technology may work together in facilitating student learning and understanding. The van Hiele theory, then, offers a means to paint with broader brush strokes, observing and seeking to explain the movement across modes of thought and styles of learning.

Although both the SOLO Taxonomy and the work of the van Hieles were inspired by the ideas of Piaget, both perspectives recognise the critical importance of the learning *context* in seeking to understand the learner. The van Hiele theory goes further still, considering the process by which appropriate teaching may contribute to cognitive development, and the critical role of language in the learner process. In this respect, the work of van Hiele has appeared to move closer to that inspired by the Soviet psychologist, Lev Semanovich Vygotsky (1962, 1978, 1987). These theories consider learner as inseparable from context, particularly the social and cultural context in which learning occurs. Vygotsky believed that all higher cognitive processes are acquired initially through social interaction - occurring on an inter-personal level before they become internalised to occur on an intra-personal level. Although he made no specific mention of the role of computer technology (since such technology did not exist in the early years of this century when his theories were developed), Vygotsky's views appear to offer much which inform a consideration of the ways in which individual learning may be enhanced by the use of suitable tools in appropriate contexts. Further developments of Vygotsky's work by Bruner (1968, 1986) and Wood (1980, 1986) provide means for detailed analysis of individual learning within collaborative and tutorial situations, particularly relevant to the present study.

Finally, as a means of describing and understanding the ways in which people learn, and ways in which people learn mathematics in particular, principles of constructivism provide a firm and broad foundation. From a constructivist perspective, learning is not achieved through the transmission of knowledge from teacher to students; rather, each student constructs his or her own meaning from the learning experiences encountered, meaning which undergoes a process of personal and social negotiation before it is internalised. Traditional exposition models of instruction assumed that the same message was received by the thirty or so different students to whom it was transmitted; constructivism denies the likelihood of such uniformity. A single instructional message may well be interpreted in many different ways. The teacher's role must still involve providing meaningful and carefully planned learning experiences; of even greater importance, however, becomes the responsibility for providing the means by which students may question what they have experienced, may compare and contrast their perceptions with others (especially their peers) and may negotiate meaning which is consistent with their existing understandings and with those intended by the teacher.

When viewed from the Piagetian perspective of constructivism, the process of learning becomes a spiral of equilibrium -> disequilibrium -> re-equilibrium. The role of the teacher, in such a view, is to attempt to induce “perturbations”, creating just enough disequilibrium in the learners to encourage re-equilibrium (Doll, 1986):

> The teacher must intentionally cause enough chaos to motivate the student to reorganise. Obviously this is a tricky task. Too much chaos will lead to disruption (Bruner, 1973, Chapter 4), while too little chaos will produce no reorganisation. (p.15)

The traditional “sequence” of instruction assumes learning to be linear and common to all learners within the group. A constructivist view, however, presents human learning as complex and branching, not simple and linear. Individuals learn in different ways: not all at the same times, nor in the same straight lines. The powerful branching features available through a program such as *HyperCard* makes such learning a very real possibility within mathematics learning situations. Students working through instructional sequences developed in this way might be expected to do so in different ways, at different rates, and to make different decisions along the way, concerning their style of investigation, and their path through such a program.

A Blending of Research Approaches

There are some who might argue that the present study is inappropriately classed as action research. Educational action research as it has been practised here in Australia has been increasingly identified with that practised by those associated with Deakin University from the early 1980s to the present time (Kemmis and McTaggart, 1988a, 1988b). Arising from a sense of frustration with "the distance between the exigencies of practice and psychometric approaches to educational research" (McTaggart, 1991b, p. 44) and wanting to "work more closely with teachers, consultants, parents and students" (McTaggart, 1991b, p. 44), their interpretation of action research borrowed heavily from critical social science in "trying to make schools and systems more reasonable, more just and more humane for students" (McTaggart, 1991b, p. 44).

The institutional focus implied in this view was quite deliberate. McTaggart notes disparagingly that "some educators working alone and following the technical imperatives of 'the action research spiral' felt they were doing action research" (McTaggart, 1991b). She goes on to state that

> as our work focused on the theoretical development of the rationale for action research... the need to specify minimum requirements for action research - the axiomatics of action research praxis - became paramount... In order to develop the rationale and purposes of action research, finer distinctions became necessary, and the language of action research became more sophisticated, and more conscious of concepts drawn from social theory (p. 44).

The present study appears by such a definition to lie outside the realm of "legitimate" action research, damned by its individual focus and its non-institutional context. The very notion of "legitimate" action research, however, appears contrary to the fundamental principles by which it was conceived, and critics of this view, such as Gore (in McTaggart, 1993a, p. 43), "claim that certain specialist discourses have occupied the action research ground and have served to disempower 'practitioners'". As Gore herself notes (Gore, 1991),

> 'Action research' as understood in teacher education circles, connotes specific practices. Given the wide range of practices that go by the name, however, it is clear that the term has no meaning outside its construction in particular discourses. If this statement is accepted, then rhetorical attempts to reserve the label for a particular set of practices ... are predestined to fail, functioning instead to police discursive boundaries (p. 47).

It is possible that the focus upon theoretical principles, the increasingly sophisticated language drawn from social theory and the perceived need to specify minimum requirements for action research described by McTaggart above may well have had the effect of distancing it from the very practitioners for whom action research was created. Denying any group ownership of a research method, the present study lays firm claim to the label, action research. While individual in focus and occurring outside a formal institutional milieu, nonetheless the primary concerns of practitioner seeking to improve his own practice through systematic application of cycles of action, observation and reflection justify such a claim.

Giving form and direction to the reflective act, and central to the analysis of data gathered for this study is the *grounded theory* method, in which the primary purpose of the research design is the generation of theory which is developed directly from the data - in this case, a theory of mathematical software use in the context of algebra learning. As defined by Strauss and Corbin (1990):

> ... (a) **grounded theory** is one that is inductively derived from the study of the phenomenon it represents. That is, it is discovered, developed, and provisionally verified through systematic data collection and analysis of data pertaining to that phenomenon. (p. 23)

The originators of the grounded theory approach defined four central criteria by which such theory may be judged (Glaser and Strauss,1967, pp. 237-250): **fit** (the extent to which the theory reflects and is faithful to the reality expressed in the data); **understanding** (the extent to which it reflects the realities of the practitioners and is congruent with their perceptions); **generality** (in the sense that the theory should be abstract enough to be applicable to a variety of contexts appropriate to the phenomenon), and **control** (being the measure of appropriate action towards the phenomenon which flows from and is directed by the theory) (Strauss and Corbin, 1990, p. 23).

The theory generated by this approach may be expected to fulfil traditional empirical requirements of significance, reliability and validity, arising as it does from a systematic analysis of the reality as reflected in the data. Through detailed and extensive analysis of both the categories by which the phenomenon may be recognised and, more importantly, the network of relationships which link these, the approach may be expected to yield results which are conceptually dense and rich in both descriptive and predictive power. Again, from Corbin and Strauss (1990):

> Its systematic techniques and procedures of analysis enable the researcher to develop a substantive theory that meets the criteria for doing 'good' science: significance, theory-observation compatibility. generalisability, reproducibility, precision, rigour and verification. (p. 31)

Lying at the heart of the Grounded Theory method is what the authors refer to as "the Paradigm Model", a method by which subcategories may be linked to a category in a set of relationships which place it into a rich and well-defined analytical context (Strauss and Corbin, 1990, p. 99). In simplified form (and within the context of the present study) this model may be represented as:

(a) CAUSAL CONDITIONS -> .... Events leading to occurrence of...

(b) PHENOMENON ->................................. Using the software...

(c) CONTEXT -> .... The algebra learning situation...

(d) INTERVENING CONDITIONS -> .... Impediments and Imperatives...

(e) ACTION/INTERACTION STRATEGIES-> Form of the interaction...

(f) CONSEQUENCES ->.............................. What are the results?

Mapping the major categories of the study in these terms forms the basis for more intensive theoretical interpretation of the data. Ideally suited to this task is the qualitative analysis software tool, *NUD•IST* (Richards and Richards, 1993) which encourages and supports the creation of logical trees and networks of relationships to form an *index system* by which the analysis may be defined. The program appears to offer an ideal complement to the grounded theory approach, supporting the detailed and yet extensive analysis demanded for the generation of substantive theory. *NUD•IST* is used in the present study to initially support the coding of data using broad general categories (*People, Words, Actions, Tools* and *Content*). These broad categories reflect the initial conception of the project on the part of the researcher. Increasingly fine detail in the coding is then facilitated by the retrieval capabilities of the program, and the “Index System” develops as a direct result of the analysis of the data. Finally, categories and sub-categories may be moved easily to reflect the growth towards a theoretical structure which results from the application of Grounded Theory procedures. More detailed description of this process is provided in Chapters Four and Five.

The study centres upon the **core category** of *mathematical software use*. It is the primary purpose of the project to describe, explore and explain this phenomenon within the context of algebra learning situations and so to learn to use these new tools as supports for effective learning. **Causal conditions** for this phenomenon will include features of the particular algebra learning situation which lead to the use (or non-use) of available software tools. These may be recognised as particular events or conditions which give rise to specific interactions between users and tools.

The **context** is concerned with the conditions under which the phenomenon manifests itself. Characteristics of the user and the learning situation are critical. **Intervening conditions** are broader, more general aspects which influence the interactions. These may serve as either *impediments* or *imperatives* (hindering or encouraging tool use).

The actual tool use may be considered in terms of **action/interaction strategies** and these taken together with the various antecedent conditions will help to determine the particular **consequences** which result. Analysis in these terms will be applied to, not only the central phenomenon, but each of the major categories and subcategories which arise from the data. In this way, the generation of theory will be both intensive and extensive in relation to the phenomenon in question.

The Computer as a Research Environment

Seymour Papert, the creator of *LOGO*, once described the computer as the “proteus of machines”, with the potential to be all things to all people (1980). While such a claim is somewhat expansive, within the domain of educational research, and particularly with regard to research of the type suggested above, the computer offers unique and exciting possibilities. When used within a flexible programming environment such as that offered by *HyperCard*, it becomes possible to use the computer as a means of studying itself - or at least the interactions of teachers and students with it.

As a means of capturing the interactions of individuals with available technology, a unique research instrument was developed for this study. Based in *HyperCard*, the program consists of three major components:

(1) A series of interactive instructional modules, spanning content and processes across the secondary years, within which computer algebra, graph plotting and tables of values tools are available, and their use encouraged. The program provides immediate access to external software tools which accompany and extend the instructional process, encouraging free exploration of the ideas and processes under consideration.

(2) In addition to the external software tools, a "mathematical toolkit" (called the *MathPalette*) is provided within the program, making available an extensive range of supportive functions which include graph plotting, tables of values, equation solving, coordinate geometry features (midpoint, gradient, distance and equation of a line through given points), numerical substitution, derivatives and areas under curves. The usual symbolic, graphical and numerical representations by which algebra learning is increasingly enhanced are supplemented within the program by an

interactive "concrete algebra" mode, by which algebraic expressions and equations may be created and acted upon using concrete representations. These facilities are intended to further support and encourage open-ended exploration of algebraic ideas and processes within the context of the instructional modules and beyond.

(3) Additional to these mathematical components of the program are specific research components, designed to generate appropriate research data related to thinking and tool use.

The "research questions" which accompany the instructional modules consist of a series of specific tasks, intended to reveal aspects of mathematical and pedagogical thinking on the parts of those using the program. These tasks range from open-ended "Grand Tour" questions (Spradley, 1980) such as *"Describe a typical maths lesson"* to the Likert-style *Constructivist Learning Environment Scale* (Taylor and Fraser, 1991). Cards portraying a range of visual algebraic images are used to generate verbal responses; the cards are then grouped by participants (after Stein, Baxter and Leinhardt, 1990). In several cases, these images are then used as the basis for a Repertory Grid analysis (after Kelly, 1955), potentially providing further insight into individual thinking and understanding of algebra. A simple attitude scale is also included, adapted from Quinlan (1992). Each of the various research components designed for the purposes of this project have been adapted to a computer-based format using *HyperCard*, allowing entry of data in a format consistent with that of the overall research design.

The "research version" of the program has been enhanced through the addition of three features, labelled "comments", "prompts" and "probes". At any time during a session, the user is encouraged to make comments regarding the program, the ideas encountered, their own responses, and so on. These comments are recorded along with the other research data from that session: which cards are viewed, which buttons are pressed, which functions are entered, graphed or analysed in other ways; times for each of these activities are also recorded. "Prompts" are questions posed within the context of the instructional module, which the user is expected to answer. They may also take the form of suggestions as to the possibilities for the use of computer algebra, or graph plotting or table of values tools at particular points. Finally, "probes" have been added at certain critical points in the modules. When users access a computer algebra tool, or graph plotter, or table of values, for example, they will be asked afterwards what their intentions were in this regard, and how effective they found the tool to be. When first selecting particular modules, such as the "Beginning Algebra" module, student teachers will be probed regarding the way in which they would sequence an introduction to algebra themselves at present; this may be compared with a similar probe at the conclusion of the module. Students are probed as to their understanding of key concepts, such as "function", "variable" and "equation", in addition to their attitudes towards mathematics, and their own assessment of their abilities in this regard. The research tool, then, provides a means, within an instructional framework, of studying the interactions of students and student teachers with mathematical software tools.

The research design is that of case study, focused upon individuals working through the instructional materials, and providing additional research data related to preferred images and representations of algebraic concepts, facility and understanding of algebraic manipulations, as well as attitudes and beliefs concerning pedagogical factors related to algebra learning. Responses are then coded and analysed using qualitative research software (*NUD•IST*) allowing regularities and relationships within the data to be observed and explained. Further, *NUD•IST* supports, not only the retrieval of data according to the codes and categories applied, but also the development of the *Index System* which defines the analysis of the data. While coding begins at a general level, it becomes increasingly refined as the data retrieved from each general code is recoded in finer detail, consistent with the grounded theory approach. Finally, these categories and sub-categories may be moved around and positioned in relation to each other, allowing the building up of a complex and intricate theoretical structure which becomes the grounded theory itself.

The grounded theory approach has been applied to the analysis of data in order to offer findings which are integrated, detailed and rich in explanatory power. The theory of *mathematical software use* which this study explores is intended to offer both detailed explanation and generalisability within the confines of the chosen research design. The overarching action research design further supports the ongoing development of a computer-based technology-rich learning environment for the learning of algebra across the secondary school years. It is anticipated that these twin outcomes of the research design will offer guidance and direction to teachers who would use mathematical software tools as means to enhance algebra learning within other settings.

Two
Review of the Literature

> First we must see how the teaching and learning of traditional topics can be improved with the full use of technology. Can these topics be taught better, faster, and with greater student understanding? When this question has been answered, curricular issues can be addressed.
>
> (Waits and Demana, 1988, p. 332)

Seymour Papert (the creator of *LOGO*) dreamed of the computer as providing the basis for a “mathland”, “which is to mathematics as France is to French, where children would learn to speak mathematics as easily and as successfully as they learn to speak their native dialect” (Papert, 1980, p. 230). Over a decade later, his dream remains far removed from the realities of the vast majority of schools and classrooms. The reasons for this lack of impact are diverse. They include concerns associated with computer technology, such as access, availability and cost, appropriateness and ease of use of both software and hardware. They include also characteristics of the teachers and learners who would use mathematical computing technology - from traditional ways of viewing and doing mathematics to the difficulties of bringing about change in schools and classrooms.

The review of the literature which follows seeks to provide an overview of issues associated with incorporating technology into the related processes of teaching and learning mathematics. In so doing, it establishes a clear direction for the course of the project which follows. The major focus in this study is the use of advanced mathematical software (particularly computer algebra software) by students and student teachers, and the interactive relationship between thinking and action in this context. In order to bring this broad area into a sharper perspective, responses of the participants will be analysed using the SOLO taxonomy (Biggs and Collis, 1991, Collis and Biggs, 1991) and the van Hiele model (1986). These methods are intended to make explicit both modes of representation of algebraic ideas and levels of thinking within an algebra learning context.

Ways in which the technology may contribute to improvements in the development of key concepts such as “algebra”, “equation”, “function” and “variable” will also be considered, particularly in the light of recent work in this area by Sfard (1991, 1992, 1994) and more established work by Bruner (in Bruner and Anglin, 1973). An overarching influence in the analysis of individual interactions with the software tools is that arising from the work of Vygotsky, particularly as it has been applied and extended to the description of collaborative and tutorial learning situations by Bruner (1968) and, more recently, Wood (1980, 1986). Since data collection for this project occurs as a result of individuals interacting with the software in tutorial or collaborative situations, these “scaffolding” studies are most appropriate.

Computer Algebra Software as a Tool for Learning

The potential for computer algebra systems to enhance the teaching and learning of mathematics has tantalised educators now for more than a decade. Some seventy articles and dissertations dealing specifically with computer algebra have been identified in twenty different journals and numerous books; the vast majority of these have been published in the past three years. Of these, however, only nineteen describe the results of research in this area and only six studies involved secondary students (Boers, 1990, Rosenberg, 1990, Sheets, 1993, Wood, 1991, Yerushalmy, 1991a, 1991b) . The increasing volume of publications in this field points to a growing interest and awareness of the possibilities for the use of such powerful tools; the lack of research evidence as to the effects of such use points further to a growing need in this regard.

Computer algebra has been available for personal computers since 1980, in the form of *muMATH*, a powerful symbol manipulator which, through innovative design, was capable of performing exact arithmetic, calculus and much of the symbolic algebra required from school through university - all using only 64 kilobytes of random-access memory! Compared to modern computers which frequently access more than 4, 000 kilobytes, this was, even by present standards, a remarkable achievement. Inexpensive (at $US40), versatile and powerful, *muMATH* stimulated enormous interest among groups of mathematics educators who saw it as potentially capable of influencing mathematics curricula away from the repetitive focus upon manipulative skills which had so characterised instruction in the past

(Fey and Heid, 1984, Fey and Good, 1985, Coxford, 1985, Ralston, 1985). This theme was pervasive through both the National Council of Teachers of Mathematics' Yearbooks in the middle of the decade, *Computers in Mathematics Education* (Hansen and Zweng, 1984) and *The Secondary School Mathematics Curriculum* (Hirsch and Zweng, 1985), and recurred again in the 1988 Yearbook, *The Ideas of Algebra, K-12* (Coxford and Shulte, 1988). *muMATH* provided the focus for numerous publications which described its potential for enhancement of such diverse areas as engineering analysis (Lance, Rand and Moon, 1985), secondary school algebra (Beard, 1989), solving differential equations (Mathews, 1989b), teaching the fundamental theorem of calculus (Mathews, 1989c) and, of course, teaching general calculus (Freese, Lounesta and Stegenga, 1986, Mathews, 1988, 1989a).

For all its power and affordability, *muMATH* has had a negligible effect upon mathematics classrooms in secondary schools. This is partly because it was not easy to use (with an enormous vocabulary of commands which required frequent reference to the manual) and it had no graphics capability; teachers also had no clear guidelines about the use of such tools. The inherent difficulties and essential inappropriateness of the program for use in schools were sufficient reasons for teachers not to persevere with its implementation.

In recent years, *muMATH* has made way for a new generation of open-ended symbolic manipulation software, equipped in most cases with such essential features as graphics capabilities, true mathematical formatting, intuitively simple command structures and extensive mathematical capabilities. Such tools are referred to here as "enhanced computer algebra tools", and include programs such as *Mathematica, Maple V*, *Derive, Theorist, MathCAD, Milo* and, here in Australia, *SymbMath*. These tools vary widely in their costs and capabilities, but all are capable of performing the majority of algebraic manipulations and algorithms required for the secondary school and beyond. Less powerful software, such as *CC - the Calculus Calculator, CoCoA* and *MathMaster* also provide access to many of the same functions as the more powerful programs of this type, but at minimal cost.

In seeking to explore the use of computer algebra software in mathematics teaching and learning, cost is but one of a number of relevant factors. The lack of impact of such tools over the past decade needs to be viewed in terms of both hardware and software constraints, in addition to issues of access and availability of computing technology in schools.

Computer algebra is extremely intensive in its memory demands. The complex algorithms and numerous calculations required for computers to carry out symbolic manipulation have generally required computing power beyond the reach of most schools. The rapid increases in both available memory and raw computing power of the past few years, however, now see affordable computers capable of operating such software. The release of low-cost computers for both Macintosh and MS-DOS operating platforms means that the hardware required for most of the existing computer algebra tools is now reaching schools and classrooms.

While memory requirements and cost of hardware are obviously significant factors with regard to the use of advanced mathematical software, they are insufficient to explain the lack of impact of the past decade, since programs such as *muMATH* has been available and affordable for this period. Another critical factor, then, lies in the nature of the software itself. As noted above, programs such as *muMATH* may be impressive mathematically, but they fall short when examined pedagogically. If computer algebra is to be accepted into secondary classrooms, then a number of critical features should be present.

The inclusion of a graphics facility has already been mentioned, and the power of the graphical representation in facilitating the learning of much of school mathematics is now well established, both by research and classroom experience (Arnold, 1991c, Demana and Waits, 1988a, Dugdale and Kibbey, 1990, Fey, 1989, Kaput, 1986, 1993, Kissane, 1995, Ruthven, 1990, Tall and Thomas, 1989, Waits and Demana, 1989a, 1989b). Care needs to be taken with the early use of graph plotting software (and the hand-held equivalents) (Demana and Waits, 1988b, Goldenberg, 1988a, 1988b); intensive studies by Goldenberg (1988a, 1988b) and Demana and Waits (1988b) demonstrate the dangers of misinterpretation of graphical information, especially in the early use of such tools. Particular problems have been found associated with inadequate labelling of axis information, leading to poor representation of scale on the part of students. Similarly, interpretation of images viewed through a restricted “viewing window” was found to cause problems among less experienced users. Students who use such tools exclusively may also fail to develop skills and understandings which appear to arise from the physical actions associated with plotting and drawing graphs (Asp, Dowsey and Stacey, 1993a, pp. 53-55).

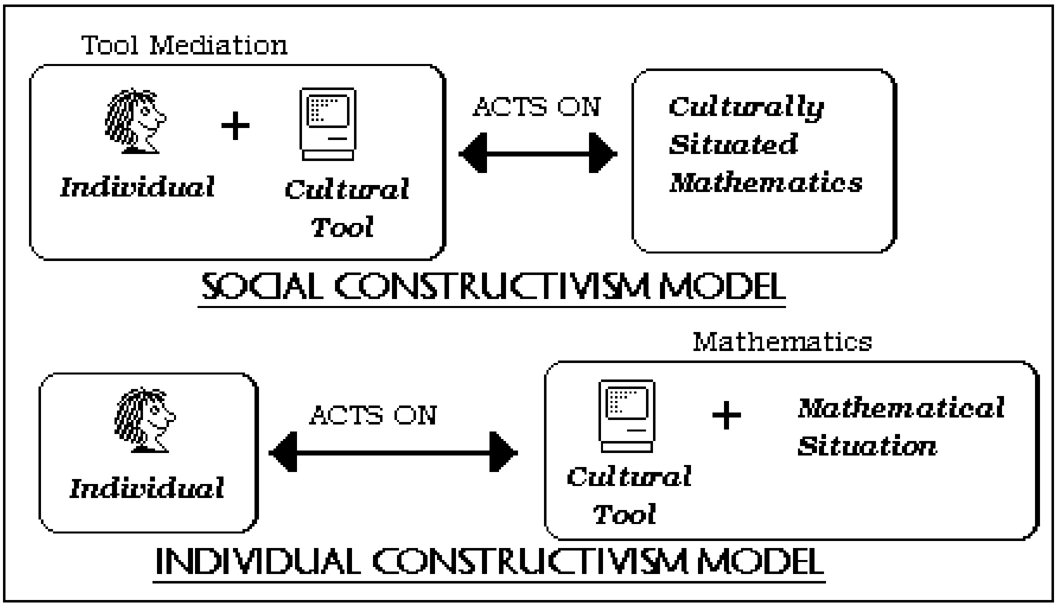
Thoughtful use of the graphical representation of functions, however, has been shown to be a powerful aid to understanding and concept development in secondary and tertiary mathematics (Goldenberg, 1988a, 1988b, Kaput, 1993, Leinhardt, Zaslavsky and Stein, 1990). In an extensive review of research within the domain of functions and graphing, Leinhardt *et al* emphasised the importance of establishing bidirectional links between the graphical and the algebraic forms of functions: “In fact, we might say that this correspondence is *the* thing to be learned” (p. 54). Additionally, the ability of a program to present tables of values for functions may also prove to be of significant advantage in encouraging students to become more versatile in their conceptions (Heid and Kunkle, 1988, Tall and Thomas, 1989, 1991). Such *multiple representation software* has been shown to encourage greater facility with problem solving and functional representations, and to lead to more robust concept development with regard to functions and variables (Afamasaga-Fuata’i, 1992, Arnold, 1992d, Borba, 1993, Harel, 1989, Schoenfeld and Arcavi, 1988, Yerushalmy, 1991b).

Another representational consideration concerns the formatting of both mathematical input and output. It is still common for students to be required to enter mathematical expressions using “computer syntax” rather than true mathematical notation. Even such powerful programs as *Mathematica* and *Maple V* suffer from this failing, although once entered, the input is converted to a reasonable semblance of mathematical formatting. Such programs also use a “command-line” format for instructing the computer: the user must type in the required command in the correct syntax (which usually involves ending with a semi-colon or some other device to signal to the computer that it is time to act). While it may be argued that there is some benefit in encouraging students to enter instructions carefully and correctly, the increased difficulties caused by this approach appear to act as a liability for use in secondary schools - especially if students and teachers are forced to memorise the commands required to operate the program. Programs such as *Theorist, Derive* and *Milo* have minimised this problem through the use of menus and templates, from which the user may select, not only the commands required, but also the notational syntax. There is no need for the user to learn arcane commands or syntactical conventions which vary from program to program, and input and output are both presented in correct mathematical form. Such a feature must be a strong consideration in choosing a program for use in the junior secondary years, and should prove useful with older students as well. In terms of this study, the choice of *Theorist Student Edition* and *Derive* as the preferred software tools was made based primarily upon ease of use and cost factors.

Closely related to the ease of use of available software tools is a sense of personal ownership on the part of the user. Smith (1992) distinguishes between two different senses in which the computer may serve a *mediating* role between the individual and the mathematics which is the object of attention. The first, which he terms the *Social Constructivist Model,* builds upon the Vygotskian notion of cognitive action within a social and cultural context. Within this framework, the image of "computer as intellectual partner" (Smith, 1992, p. 15) is illustrative of a particular mediating role in which child and computer together act upon the mathematics which is “out there”. This culturally situated mathematics also impacts upon the user in a cyclic way (Smith, 1992, p. 18). Distinct from this model is that which Smith designates the *Individual Constructivist Model,* in which the computer, like the mathematical situation, is perceived as being external to the user - the computer is something to be *acted upon*, just as is the mathematics (see Figure 2.1). Both are *external* to the user, whereas within the Social Constructivist framework, the computer is a *personal* tool. Dick (1992) makes a similar connection involving student use of hand-held calculators which, by reason of size and personal ownership are more readily accepted as tools.

This distinction between ways in which the computer is perceived and used by individuals is seen as critical to the current study, and provides a significant area of focus within the research design.

Figure 2.1: Smith’s (1992) distinction between personal and external computer use



Research on Computer Algebra

The earliest empirical study of the effects of the use of computer algebra in the teaching and learning process would appear to be that by Heid (1988), which was based upon her doctoral study completed in 1984. In this quasi-experimental study, Heid compared two groups of college students (n = 39) studying applied calculus, using traditional methods with one group, while the other used graph plotting and symbol manipulation software (*muMATH*) to perform routine manipulations. While the control group spent the full twelve weeks of the course practising the manipulative skills required in the course, the experimental group spent only the last three weeks on skill development. Prior to that, all computations were performed using the computer, while the instructional focus was upon concept development and understanding. Heid found that, while the two groups performed equally well on tests of manipulative ability, the experimental group demonstrated better understanding of the required concepts.

Similar results have been found in other studies comparing groups using computer algebra with those following traditional approaches (Boers, 1990, 1992, Palmiter, 1991). The study by Palmiter used a relatively large group (n = 78) of university calculus students as the sample, and demonstrated not only improved conceptual knowledge as a result of the use of the program *MACSYMA*, but also better computational scores using the computer algebra system than other students using pencil and paper methods (Palmiter, 1991, pp. 153-156).

Boers (1990) investigated the acquisition of the concept of variable in a traditional as opposed to a "computer-intensive" algebra curriculum, in which the program *Derive* was used by a middle-school class of Algebra 1 students to perform all algebraic manipulations. Six students from each group were interviewed twice during the academic year, and the results indicated that the use of computer algebra encouraged in students an understanding of variables as "generalisers" and as "varying quantities that are dependent on other quantities" (Boers, 1992, p. 6), as opposed to the more static concept of variable as an "unknown quantity" which appeared to result from the traditional approach. The experimental group demonstrated also a greater ability to model problem situations and to read and interpret tables of values and graphs than their traditionally instructed peers (Boers, 1992, pp. 4-5). Similar improvements in conceptual understanding without corresponding loss of manipulative facility were supported by other studies involving secondary students using computer algebra systems as scaffolding tools (Rosenberg, 1990, Wood, 1991).

The program *Maple* has been used with both Business Calculus and Calculus 1 students in studies which support the earlier findings by Heid and Palmiter that skill acquisition is not a necessary prerequisite to conceptual understanding or the ability to apply calculus to problem solving situations (Judson, 1990a, 1990b).

These tertiary studies of the use of computer algebra tools differ from secondary school studies in that they involve students of mathematics who, however diverse, must nonetheless be considered select groups of better than average mathematical ability. The secondary studies cited were not concerned with the acquisition of algebraic skills but with the relative understandings associated with particular central concepts. Questions regarding the effects of the use of computer algebra upon skill acquisition among those beginning their studies of algebra remain as yet largely unaddressed by research.

Studies by Yerushalmy have investigated the effects upon junior secondary students of the use of specifically designed tools which combine algebraic manipulation capabilities with graph plotting in what he terms “multiple representation software” (Yerushalmy, 1991a, 1991b). The first study explored the effects of computerised feedback on the ways in which a group of 25 seventh-grade students carried out algebraic transformations, and their approaches to “debugging” their own processes. They were divided into four groups - one which received no feedback regarding their work, another which was simply informed as to whether their results were correct or incorrect; the other groups had computer assistance to provide feedback - one in graphical form, and the other using a symbol manipulator. Students who received feedback on their work in graphical form were found to be motivated to carry out further investigations upon their work - they were able to see the graph of the desired end-product of an algebraic transformation, and the graph of their own result, and were encouraged to act upon their work in order to “correct” their errors. The “manipulator” group was found to be motivated to arrive at the “correct” result using the tools provided. The group given “judgemental” feedback (correct or incorrect) performed better than that with no feedback at all on their end results, but still lacked motivation and probably the ability to correctly complete the tasks. The implications for teaching from this study suggest that computerised feedback in both symbolic and graphical forms can serve as a strongly motivational factor with regard to students performing algebraic transformations, and can encourage them to persevere and continue to explore given tasks to a greater extent than students without such feedback.

Yerushalmy found in another study that software which encouraged students to explore "multiple representations" of functions (symbolic, numerical and graphical) enhanced their understanding of the concept of function and encouraged a more versatile approach to problem solving. This study, with 35 eighth graders, also suggested, however, that the cognitive "links" between the various representations were fragile, and did not occur spontaneously (Yerushalmy, 1991b, pp. 54-55). Student misinterpretations and informal theories were common, as was the tendency to view functions as objects rather than processes. There is a need to implement the use of such powerful computer tools thoughtfully and with caution (Yerushalmy, 1991b):

> During this course the students developed and continued to use theories which reflected both their correct understandings and their misconceptions. As Goldenberg (1987) claims, such theories might not be efficient, but they have their own value. As it was shown, these theories do not impair student ability to carry out correct techniques in an efficient way. (p. 55)

There exists, then, a growing body of evidence which suggests the benefits of the use of computer algebra software as an aid to mathematics learning and instruction in secondary schools. Used carefully, such tools may be expected to contribute to improvements in concept development and understanding, in addition to more positive attitudes towards the subject. Students using such aids have been found to be more effective and persistent problem solvers, and more inventive in their approaches. No evidence has been found to support claims of reduced manipulative ability when such software is used to support computation.

The use of computer algebra, however, is not without its critics. In particular, Waits and Demana (1988) argue that little is to be gained from “a device that simply symbolically produces the answer” (Waits and Demana, 1988, p. 334). They argue the case strongly for the use of graph plotters and particularly graphics calculators as essential aids to mathematics instruction. In dismissing the use of computer algebra, however, they suggest the following as reasons (after Waits and Demana, 1992):

- Cost and accessibility preclude the majority of students from using such technology, both at home and school;
- ... (E)xact answers produced by computer symbol manipulators are often of no real use and sometimes furnish little insight into the problem modelled by the algebraic representations.
- ...(S)tandardised tests change slowly and students will be required to demonstrate some ability in algebraic manipulation on them for some time to come...
- ... (N)o one can be sure at this time how much paper-and-pencil algebraic manipulation is really necessary for success in college and in a work place that requires increasing technological and scientific know-how. (p. 180)

While such arguments provide valuable cautionary guidelines from which to view the use of computer algebra software in schools, they fail to adequately argue against their introduction. Many of the same criticisms regarding access and affordability have been levelled against graphics calculators in the past, prior to their reductions in price in the past two years. The same reductions and increased availability may be expected to apply to computer algebra tools, with “free” and “shareware” programs such as *MathMaster* and *CoCoA* for the Macintosh, and *CC3 - the Calculus Calculator* for MS-DOS machines already available. The problem of student access to hardware, of course, remains critical.

The issue of *personal access* to classroom computer technology for the students (discussed above) may be directly related to the development of hand-held mathematical computers which offer graph plotting and, in some cases, symbol manipulation (Ruthven, 1992, Dick, 1992). “Portable” computer algebra systems such as the Hewlett Packard HP-28 series have been available for some years, and present another alternative to the problems of access and equity which have already been mentioned. Although expensive in comparison with numerical calculators and even graph plotters, the cost of a class set of such tools (eight to ten units, allowing senior students to work in groups of two or three) is comparable to that of a classroom computer. Research on the use of such tools in senior classes suggests that they can contribute to improvements in concept development and understanding, student attitudes and confidence (Arnold 1990, 1992e). The symbolic manipulation capabilities of these calculators are, at present, limited - they tend to be slow and difficult for students to master, and so do not provide a sufficiently transparent and intuitive tool for use across the secondary years. The advantages of portability and personal access, however, make their use in the senior years attractive.

The argument that exact answers to symbolic problems actually obscure interpretation of the solution (Waits and Demana, 1992, p. 181) is an interesting one; however, most computer algebra tools allow answers to be expressed in both exact and rational approximation forms, a choice not available on hand-held computers. Similarly, criticising computer algebra systems because they sometimes produce different forms for the same answer seems also to be somewhat short-sighted, since this provides a powerful incentive and opportunity for student exploration of mathematical identities, and certainly a valuable basis for classroom discussion.

In summary, then, the literature provides adequate evidence of the potential for computer algebra tools to enhance the teaching and learning of mathematics at all levels from the junior years of secondary school to the tertiary years. While the current state of the research in this area remains far from conclusive, and there is evidence that there are dangers and pitfalls in the use of such technology, there appears to be much to recommend further research on the effects and implications of such instructional practice. In the absence of extensive research evidence with regard to this new and expanding educational field, the value of further research is heightened. As increasing numbers of educators begin to explore the classroom implications of advanced mathematical software, their observations and opinions provide valuable guidelines for others entering the same largely uncharted waters.

The Contribution of Vygotsky

Both Piaget and Vygotsky began their studies in the early years of this century. While Piaget's life and work spanned most of this century, Vygotsky died in 1934, at only 38 years of age. His work was largely unknown outside his own country until the 1960s; even within the Soviet Union, it was suppressed for many years. Over the past three decades, however, Vygotsky's writings have assumed growing importance, particularly in the development of wholistic theories of language learning, but increasingly in fields such as mathematics learning and teaching (Zepp, 1989, Manning and Payne, 1993, Confrey, 1993b). His notion of a *Zone of Proximal Development* has proved appealing in a wide variety of contexts, and offers much in the present exploration of the way in which appropriate computer tools may assist the growth of understanding.

In seeking to understand something of the contribution of Vygotsky in the present context, it is appropriate to begin with the notion of "tools". He began his work, *Thought and Language* (Vygotsky, 1962) with a quotation (in Latin) from Sir Francis Bacon, translated by Bruner (1986) as:

> Neither hand nor mind alone , left to itself, would amount to much. And what are these prosthetic devices that perfect them? (p. 72)

The additional tools to which Vygotsky appears to be referring are, most importantly, thought and language, those means by which we are recognised as most uniquely human. In opposition to Piaget, Vygotsky places language as the precursor to thought, claiming that it is only through the use of language that the higher mental processes may develop and become operational. Language is social in origin, developed through interaction with others, and, in Vygotsky's view, serves two primary purposes - self-direction and communication. This perception of language as a tool which aids thought is a fundamental feature of

Vygotsky's view. He believed that the higher mental processes are *mediated* by language, first observed as egocentric speech in children, which then becomes internalised, developing into thought. (Zepp, 1989, p. 30-32)

Mathematics shares with language these twin characteristics: it is, at once, both cognitive tool and means of communication. This distinction (drawn by Confrey, 1993b, p. 50) is significant in the context of the use of computer technology in mathematics learning. In its role as tool, mathematics may be perceived primarily as a means of effecting some outcome; as Confrey points out, the image of mathematics as *tool* links it with *action*, a significant aspect often overlooked. The potential for computer technology to assume an active mediating role in supporting mathematical thinking, learning and practise, is critically important in the current study. The potential role of goal-directed action may be exemplified by comparing the interface offered by the *Macintosh* computer algebra package, *Theorist*, with that of other packages of the same type. Most computer algebra tools allow the immediate solution of equations, for example, through a general "Solve" command; in some cases, the intermediate steps of the equation-solving process are supported by allowing operations to be carried out on both sides of the equation. *Theorist* is unique in allowing the user to physically manipulate the terms and elements of algebraic expressions and equations. Solving an equation such as

$$\frac{3}{x - 1} = 2$$

may be achieved by physically selecting the denominator, **x - 1**, and, using the mouse, dragging it across to the right-hand side of the equation, to automatically produce

$$3 = 2(x - 1)$$

This may be expanded, and the **x** term isolated by similar manipulations. The program offers the choice of alternative techniques which, in some ways, may be preferable pedagogically (involving performing the same operation to both sides of the equation, producing equivalent equations). At the same time, many students and teachers solve equations in exactly this way, involving either overt or covert physical manipulation of the terms. The role of action in higher mathematical processes is likely to be significant, but as yet remains largely unexplored. This recent development of computer software which simulates and supports such physical involvement invites such exploration.

In the context of the present study, the thinking of individuals as they interact with advanced mathematical software is seen to be accessible through the twin avenues of *action* and *language*. The software tool designed for the study captures both aspects of the interactive process. It records the actions of the users as they engage in a wide range of mathematical tasks, both independently and within the context of the instructional modules provided - each button that is pressed, each option chosen, the time spent at each point of the process; these important elements of physical involvement become part of the record of interaction which the software provides. Written comments in response to questions, prompts and probes generated by the software throughout the session also become part of this record, allowing the language of the user to be coupled with the concurrent interactions.

Central to Vygotsky's view of cognition and learning is the social and cultural context of the learner. In particular, the learner is viewed as achieving higher cognitive ground through interaction with others, especially adults and knowledgeable peers. His *zone of proximal development* may be thought of as “the distance between the actual developmental level as determined through independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers” (Vygotsky, 1978, p. 86). Bruner explores the implications of this concept for tutoring and guided learning through his notion of “scaffolding” (Bruner, 1986, pp. 74-76). Since the use of the software by students in the present study occurs most often within such a context, these ideas are particularly significant.

Wood (1980, 1986) expands upon both Vygotsky’s zone of proximal development and Bruner’s notion of scaffolding to offer a model of learning based upon twin central principles of *uncertainty* and *contingency*. Wood observes that learning within a situation of uncertainty is always less effective than one in which the learner is able to recognise commonalities and familiar features. Motivation, task orientation, even the ability to remember particular features of the situation - all are likely to be reduced in unfamiliar situations, and much mathematics learning occurs within situations high in uncertainty. Support is required, then, in such situations of high uncertainty which will serve to alleviate these problems and so make the learning experience more effective.

Wood’s second key principle defines the preferred nature of such support, requiring that the response of the tutor be *contingent* upon that of the child if optimal cognitive progression is to occur. Wood postulates five levels of increasing control which may be observed in a tutorial situation (that is, a learning situation involving interaction between the learner and a more capable other - the "tutor") (Wood, 1986, pp. 197-198). These range from minimal control - the tutor prompts the learner with a general question, such as "What might be done here?" - to highly controlled, in which the tutor actually demonstrates the steps needed to fulfil the requirements of the task. Wood's principle of contingency requires the tutor to decrease the level of control at each correct action of the learner, and to increase the level of control or intervention upon each error. This process of flexible scaffolding allows the learner to progress optimally across the zone of proximal development, with greater and lesser degrees of support as required. The critical principle in such learning is the promotion of autonomy and independence on the part of the learner. It is not difficult to support and scaffold learning; the challenge lies in doing this in such a way that the scaffolding is gradually removed, and the learner actually decreases the level of dependence upon the support structure as the learning sequence progresses. This is the primary goal of contingent learning. In the context of mother-child instructional situations, Wood cites research which supports such learning: "What we find is that the more frequently contingent a teacher is the more the child can do alone after instruction" (Wood, 1986, p. 198).

A common assumption has been that the necessary support must be given by others; the potential role of the computer in such a relationship remains open. In particular, the use of software which supports the manipulations and representations of high school algebra appears to coincide with this notion of scaffolding. Certainly, if used to complement an adult-child tutoring relationship all requirements are fulfilled. At the same time, two other scenarios may be considered: the first involves two peers, both beginning their study of some aspect of algebra, with access to computer algebra software. Although neither may serve the role of “expert”, their social interaction and verbalisation while working together using the computer as support (effectively assuming the “expert” role) appears to offer much in common with the Vygotskian notion. Certainly, the situation offers the opportunity for the verbal and social interaction so necessary for the achievement of cognitive progression, while the software offers the means of navigating across the zone of proximal development, allowing work in advance of the current state of both students.

Another scenario, too, may be considered. In this situation, the role of “expert” is again taken by the computer, but this time in the form of interactive instructional material prepared in advance. The individual student may work through such materials supported by access to suitable advanced mathematical software which will aid and encourage enquiry and exploration. The student enters responses and comments, and answers questions as they occur in the work, forcing verbalisation of the developing ideas. This model, too, shares much in common with Vygotsky’s model of learning, in which the learner is challenged to move beyond the present level, and is supported in this movement by access to appropriate software. The *Exploring Algebra* package (Arnold, 1993) has been developed using *HyperCard* on the *Macintosh* computer to provide just such a learning model, especially when used in conjunction with an appropriate support program, such as *Theorist*

Student Edition.

The structure of the learning environment is the focus for research by Valsiner (in Rogoff and Wertsch, 1984) which extends the study of the zone of proximal development. Exemplified by the adult-child learning experiences associated with the socialization of meals, Valsiner proposes two additional zones which serve to define more clearly some of the situational constraints which may act to support or impede progress across the zone of proximal development. The first of these constructs, called the *zone of free movement* (or ZFM) is based upon the observation that learning is facilitated by focussing the attention of the learner upon that which is to be achieved. This may be done by restricting the actions of the learner, or by defining a “zone of free movement” (Valsiner, 1984).

> Within the field of objects and affordances related to them in the environment of the child, the zone of free movement (ZFM) is defined for the child’s activities. The ZFM structures the child’s access to different areas of the environment, to different objects within these areas, and to different ways of acting upon these objects. (pp. 67-68)

In the present study, this zone is defined by the boundaries of the software, with the *HyperCard* modules serving as the base from which other tools may be readily accessed, and then returning once again to continue with the task at hand. The parameters of the ZFM, then, are clearly defined within the context of the computer tools available.

Defined conjointly with the zone of free movement is a *zone of promoted action* (ZPA). If the ZFM is effectively an “inhibitory mechanism” (Valsiner, 1984, p. 68) which functions to limit the actions of the learner within the structured environment, then within that zone exist "sub-zones" which are defined by those actions sought to be encouraged and learned. In the context of "meal time", these may involve the appropriate use of cutlery; in relation to the present study, the zone of promoted action will be defined by the appropriate use of available software tools to achieve mathematical goals. In particular, the ready accessibility of computer algebra, graph plotting and table of values utilities encourage their use by the learners; the extent and form of such use becomes the primary focus of this study.

Theory based upon the work of Vygotsky, then, offers much which may inform and direct a study of the use of advanced mathematical software. Such theories provide significant guidance in the search for ways in which such software may be used, and such use studied. There is a need, however, for more descriptive detail if the interactions of students and teachers with the technology are to be made explicit. In particular, Vygotsky recognises the existence of qualitatively different levels of development and styles of thought, but does not pursue or expand on these. If the algebraic thinking and understanding, and the thinking about teaching and learning, of both students and teachers are to be observed and monitored, then such detail is a necessity. In the domain of mathematical thinking, the theory of Pierre van Hiele and Dina van Hiele-Geldof offers a suitable framework.

A Theory of Mathematics Education

Although van Hiele's theory (van Hiele, 1986) has been most widely recognised for its role in explicating the levels of thinking associated with the learning of geometry, it has been developed as a general theory of mathematics education. Growing from the concerns of teachers, it does not stop at the description of "levels of thinking", but seeks to provide a basis for understanding the movement between these levels, and the role of the teacher in assisting such progression. As such, the theory goes beyond the SOLO taxonomy (which is essentially descriptive) and beyond, too, the concerns of Piaget, who deliberately distanced himself from the question of how students may be encouraged to progress from level to level. His was a developmental theory, holding that such progression was largely independent of the influence of instruction; he referred to such concerns disparagingly as "the American question", but in fact it was the Dutch van Hieles who appear to have made significant progress in addressing it.

In his recent work (1986), Pierre van Hiele describes a theory of mathematics education arising from the study of two fundamental concepts - structure and insight. Although reluctant to specify a definition for the first, van Hiele admits that it may be broadly thought of as a "network of relations" (van Hiele, 1986, p. 49) in which commonalities are recognised across all types of events and perceptions. In everyday life, we recognise structure in going through daily routines, at work and home; structures are apparent in the patterns of nature and man; continuing a sequence of numbers is a recognition of structure, as is the recognition that the symbol $\mathbf{(x + 2)^2}$ may be seen as a sign to expand the given expression and produce a new equivalent one. *Insight*, in this context, is a recognition of *structure* - we know what to do when we experience such insight, and it is precisely an absence of such insight which leaves so many school students at a loss as to know what to do with a given algebraic expression, equation or problem.

Van Hiele distinguishes between *rigid* and *feeble* structures (van Hiele, 1986, pp. 19-23), strongly reminiscent of Wood's principle of *uncertainty* (Wood, 1986). Consider, for example, a student presented with the expression, $\mathbf{x^2 - (x + h)^2}$. The more likely response for a student of at least moderate algebraic facility is to attempt to expand and simplify the expression. The recognition of the requirement to expand the squared part of the expression may be thought of as a relatively rigid structure. The recognition that such an expression provides an opportunity for factorisation, as a "difference of two squares", however, is likely in most students, to be a relatively feeble structure. Some prompting may be required for students to recognise this structure, even when they quickly recognise it in a case such as $\mathbf{x^2 - 4}$. The dominant strategy of algebra instruction in the past has centred around the development of rigid structures through repetition, seeking to "automate" student responses to algebraic prompts. Such learning, however, is likely to occur at a very superficial level, and is relatively easily exposed when students encounter an exceptional case. As explained by Confrey, this relates closely to the Vygotskian notion of "pseudoconcept" (Confrey, 1993b),

> ... acknowledging that children often use words before they have grounded its [sic] meaning in conceptual operations. Vygotsky suggests that this use of language that runs ahead of cognitive depth is an important part of learning - and describes a key mechanism in how adults teach children to advance to higher levels of cognitive thought. (p. 50)

This recognition of the central role of language in the learning process is a common theme throughout the works of both Vygotsky and van Hiele.

To van Hiele, true learning is that which students achieve through their own efforts, efforts which involve them in experiencing what he terms a "crisis of thinking" (van Hiele, 1986, p. 43). Similar to the Piagetian notion of disequilibrium, and very close to Doll's "perturbation" (Doll, 1986, p. 15), van Hiele sees such a crisis as necessary for students to achieve a higher level of thinking. While teachers may be successful in having students "mimic" the responses of a higher level, unless the learner has struggled with the material personally, no cognitive gain has been made. The cognitive "safety nets" (described by Tobin and Fraser, 1988) which are a feature of many mathematics classrooms are attempts by students (and by their teachers) to reduce the cognitive load of the material to be learnt; such efforts in van Hiele's view, must be carefully controlled, since meaningful learning involves transition to a higher level of thinking, and this can only occur by going beyond the present state. The links with Vygotsky's zone of proximal development are apparent, where "the only good learning is that which is in advance of development" (Vygotsky, 1987, p. 89).

This is the point at which the theories of learning described here coincide. For all their various forms and distinct priorities, the common ground is the perceived need for *challenge*. The teacher does not encourage learning by predigesting the material; rather, the learner must be an active participant in the process of creating meaning through interacting with that which is to be learnt in a context which supports exploration, verbalisation and activity.

The ways in which the *levels of thinking* proposed in the van Hiele theory (van Hiele, 1986, p. 53) complement those of the SOLO taxonomy have been described in detail elsewhere (Pegg, 1992a). The van Hiele levels begin, not with the level of *action* proposed as the sensori-motor mode of the SOLO taxonomy, but with the level of *visualisation* or *recognition* (Hoffer, 1981), corresponding to the global, intuitive thinking associated with *ikonic* thought. Next is the level of analysis, or the *descriptive level* (van Hiele, 1986, p. 53), corresponding closely to the concrete-symbolic mode of the SOLO taxonomy. This is followed by a level alternatively labelled *abstraction* (Burger and Shaughnessy, 1986), *ordering* (Hoffer, 1981, p. 14) or, simply, the *theoretical level* (van Hiele, 1986), easily recognised as encompassing *formal* modes of thought. Although the literature describes successive levels (commonly as *deduction* and *rigour*) van Hiele himself appears more inclined to view these as logical extensions of the *theoretical level* (van Hiele, 1986, p. 53) which, once achieved, experience a phenomenon he describes as *level reduction* (van Hiele, 1986, p. 53). Although the objects of thought may involve successively higher levels of abstraction, the actual mode or style of thinking remains essentially the same. The correspondences which occur with the SOLO taxonomy enable the two models to be considered as logically compatible; the different ways in which each illuminates each style of thought, however, makes the synthesis proposed here attractive.

The theory of van Hiele, in addition to describing levels of thinking, offers an important addition. This is the notion of *stages of learning* as means by which the learner may be assisted to seek higher cognitive ground. Five such stages are specified (van Hiele, 1986):

1. In the first stage, that of *information,* pupils get acquainted with the working domain.

2. In the second stage, that of *guided orientation,* they are guided by tasks (given by the teacher, or made by themselves) with different relations of the network to be formed.

3. In the third stage, that of *explicitation,* they become conscious of the relations, they try to express them in words, they learn the technical language of the subject matter.

4. In the fourth stage, that of *free orientation,* they learn by general tasks to find their own way in the network of relations.

5. In the fifth stage, that of *integration*, they build an overview of all they have learned of the subject, of the newly formed network of relations now at their disposal. (pp. 53-54)

These stages of learning are significant in providing a framework for instruction aimed to develop understanding of the material or skills to be learnt. Each of the five stages relates to an aspect of the *HyperCard* program, *Exploring Algebra* (Arnold, 1993) developed for this study of the ways in which teachers and students interact with advanced mathematical tools. The program was designed to provide information about the topic to be studied, using the point and click interface of the *Macintosh* and the branching features offered by *HyperCard* (Stage 1). Problems are posed within the materials (Stage 2), and computer tools made easily accessible by which such problems might be investigated (Stage 4). Although a "comment" option is provided, it remains for the teacher or tutor to pursue the verbalisation required for stage 3, and to draw together the materials for the last stage of integration. Probes and prompts (described below) seek to elicit responses concerning understanding of the material and thinking about the critical concepts of algebra (such as equations, functions and expressions), but the student should ideally share their thinking with another at some stage in the process. This illustrates a further consistency with the Vygotskian notion of the zone of proximal development.

The SOLO Taxonomy

Both pedagogical and mathematical thinking may be viewed as consisting of a range of elements, operating on different levels. The SOLO taxonomy provides valuable insights into the nature of such elements. Building upon the developmental learning theories of Piaget and Bruner, Biggs and Collis recognised that learners demonstrate distinct modes of functioning, generally corresponding to the following age periods:

From Birth .... *Sensori-Motor*
From around 18 months. *Ikonic*
From around 6 years ...... *Concrete-symbolic* (approx. K-Year 10)
From around 16 years.... *Formal* (approx. Years 11 and 12 +)
From around 20 years.... *Post-Formal* (University/professional practice)

Differing from classical stage theory, it is not suggested that each stage *replaces* the previous one, but that each adds to the available cognitive repertoire. In different situations, learners may "regress" to an earlier acquired mode of functioning or utilise a higher cognitive function in the learning of a lower-order one, adopting a "multi-modal" approach to the task at hand (Biggs and Collis, 1991, Collis and Biggs, 1991). An example of the first situation (labelled "top-down" learning by the authors) would be the use of intuitive, visual methods in mathematical problem solving, where the *ikonic* mode is used to supplement the more usual *concrete-symbolic* approach. "Bottom-Up" learning may be illustrated by the use of higher-order practices in the learning of sensori-motor skills (such as thinking through the action of a golf swing, studying the style of expert players or learning the theory and techniques of art in order to improve in the ikonic aspects). Although much of secondary schooling may be recognised as occurring within the *concrete-symbolic* mode, the use of multi-modal strategies may be more extensive than previously realised (Biggs and Collis, 1991, Collis and Biggs, 1991). It is certainly common in areas such as music, which utilises sensori-motor, ikonic and concrete-symbolic elements in the learning process, and recent research suggests that the ikonic mode may be a powerful influence in mathematical problem solving (Collis, Watson and Campbell, 1992). At present, however, much of the focus of instruction in secondary schools lies within the *concrete-symbolic* domain. Even at the senior level, it is now believed that the end-point of instruction in most subjects will be at this level. Only in those areas in which the student is particularly competent (and likely to continue into tertiary study) is *formal* mode functioning likely to be observed with any degree of frequency (Collis and Biggs, 1992).

Some learners never reach the *formal* stage, at which the foci of interaction are theories and abstractions, rather than the more concrete objects of earlier stages; many, perhaps most, do not achieve *post-formal*, which involves working with and extending theory *systems* themselves. With increased retention rates in the senior years of schooling, it is likely that increasing numbers of senior students will be operating throughout their studies at the *concrete-symbolic* level (Collis and Biggs, 1983). The preferred mode of operation for students has significant implications for learning and instruction (Collis and Biggs, 1991), and will become a focus for this investigation in the study of the representation and understanding by students and preservice teachers of algebra and learning interactions.

Further, Biggs and Collis suggest that, *within* each mode of functioning, learners display a consistent sequence or "learning cycle" when learning new tasks. This gives rise to the SOLO acronym, detailing the "Structure of the Observed Learning Outcomes". The theory postulates five distinct levels or outcome structures (from Biggs and Collis, 1989):

Prestructural	The task is engaged, but the learner is distracted or misled by an irrelevant aspect belonging to a previous stage or mode.
Unistructural	The learner focuses on the relevant domain, and picks one aspect to work with.
Multistructural	The learner picks up more and more relevant or correct features, but does not integrate them.
Relational	The learner now integrates the parts with each other, so that the whole has a coherent structure and meaning.
Extended Abstract	The learner now generalises the structure to take in new and more abstract features, representing a higher mode of operation. (p. 152)

Using this framework it becomes possible to identify an individual's current level of operation for a particular task through a study of verbal and/or written responses. It thus provides a powerful tool for the assessment of student understanding of concepts, and for problem solving (Collis and Romberg, 1991). The taxonomy has also proved effective as a means of planning and developing curricula based on the cognitive characteristics of the learners (Stanbridge, 1990).

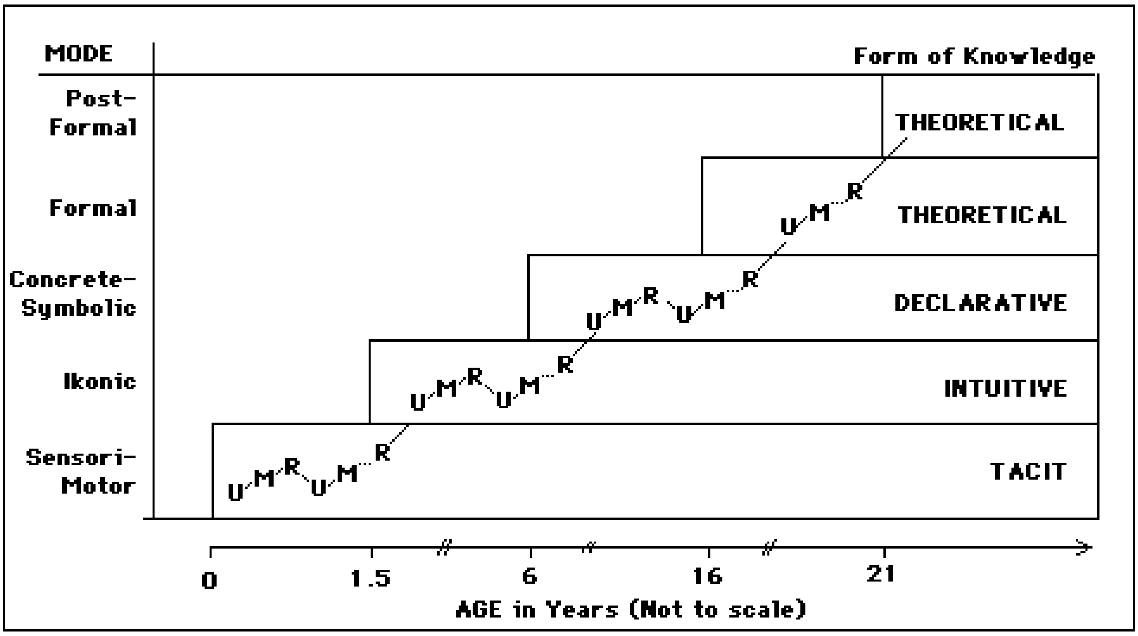
In terms of pedagogy, the unistructural, multistructural and relational levels are recognised as the “target modes” for teaching; allowing for individual differences, it may be expected that all students should achieve one of these levels as a result of an effective learning experience. In the case of new work, it should be the teacher’s objective to assist the students to move from a prestructural state (with no organised or coherent knowledge of the material) to one that is, ideally, relational. In practice, however, the more likely end result for instruction is *multistructural*, in which students and their teachers are satisfied to know “some things about an area”. Relational understanding (in both the SOLO sense and that of Skemp, 1976) is frequently sacrificed for the demands of utility. The occurrence of an extended abstract response is not normally one that is anticipated as a direct result of instruction, but is more a function of the individual learner’s ability to go beyond what has been taught.

Recent research suggests that, just as the linear sequence of modal development originally proposed has given way to a more complex multi-modal structure, so the cycle of levels within the modes may be more complex than originally anticipated. In particular, at least two Unistructural-Multistructural-Relational cycles now appear to exist within the *concrete-symbolic* mode, as observed across a range of mathematical topics in the junior years of high school (Pegg, 1992b).

This would help to explain quite distinct styles of thinking about complex mathematical objects (such as process and object conceptions of functions) while situated within a single cognitive domain. As more is revealed through research, the model of cognitive development offered by the SOLO Taxonomy assumes less of the linear, sequential pattern of its Piagetian origins, and more of a complex branching structure.

Figure 2.2 (Pegg, 1992a, p. 27) provides a schematic outline which relates modes, learning cycle, curriculum goals and suggested exit levels for schooling. Each mode of operation is associated with a particular "type of knowledge", as illustrated. That arising from the *sensori-motor* is likely to be *tacit*, unable to be articulated, as in the "feel" of a good golf swing. The *ikonic* mode produces knowledge which is *intuitive*, difficult to verbalise, and closely linked to visual and emotive aspects of the situation. The *concrete-symbolic* mode leads to knowledge that is *declarative* - not only knowing "how to", as in earlier modes, but being able to say "why", at least in terms of the concrete referents available. The *theoretical* knowledge which results from the later modes involves adopting theories and theory systems - complex networks of relations between ideas and concepts - as the objects of thinking.

Figure 2.2: SOLO Taxonomy: Schematic Outline



Images and Definitions of Algebra

The concept of "function" is increasingly recognised as central to an understanding of algebra across the years of secondary schooling and beyond, particularly within a technology-rich context (Harel and Dubinsky 1992, Grouws, 1992, Romberg, Fennema and Carpenter, 1993). Senior students are expected to be familiar with a range of common functions, including the linear, quadratic, trigonometric, exponential and logarithmic functions; to sketch, manipulate, differentiate and integrate them. The study of functions occupies the major part of the time spent on mathematics in the senior school.

The concept of "function", not surprisingly, is one which is mathematically rich, capable of being thought of using a number of distinct representations, or "images". Numerous studies over the past decade have investigated the ways in which different groups of people think about and use functions. Barnes (1988, p. 121), in interviews with secondary and tertiary students in New South Wales, identified six frequently occurring images of functions:

- A graph or curve
- A set of ordered pairs or table of values
- A relationship between two variables
- An algebraic formula or equation
- A "function machine" (input-output device)
- The symbol f(x)

She also distinguished the image of function as a "*mapping between two sets*", which was not widely used by students, but was seen as useful in thinking about the concept.

Barnes interviewed what she described as a "small group of Year 11 students, who had recently begun calculus" (Barnes, 1988, p. 119). She found that the dominant image of function used by these students was a *graphical* one and that there was widespread uncertainty about "what a function is" (Barnes, 1988, p. 122) - presumably the more formal definition of the concept. A more recent and extensive study, which was undertaken to provide baseline data for the current project, found a different pattern of representation (Arnold, 1992d). This study, of close to 400 high ability secondary and tertiary mathematics students found that these students were most likely to think of functions as algebraic formulae or equations, and slightly less likely to use a graphical image, or to think of them as rules or relationships between variables.

Studies by Vinner and his colleagues over the past ten years (Vinner, 1983, Tall and Vinner, 1989, Vinner and Dreyfuss, 1989) distinguish between a “concept image” (“the set of all the mental pictures associated . . with the concept name, together with all the properties characterising them” (Vinner and Dreyfuss, 1989, p. 356)) and a “concept definition” (“a verbal definition that accurately explains the concept in a non-circular way” (Vinner, 1983, p. 293)). Such studies reveal, among other things, that concept images may not always be consistent with the formal definition, but that such inconsistencies are often not apparent. Much of the focus in this area has been upon identifying individual images which students prefer to use when thinking about functions, although the verbal descriptions given as definitions of functions frequently comprise multiple images. This was further supported in Arnold (1992d), in which the pattern of representation was quite different when students were asked to describe a function “in their own words”. In this situation, functions were most likely to be described as a “rule or relationship”, which coincides with the common (non-mathematical) idea of “function”, or one of several “multiple-image” definitions, such as an “algebraic object which can be graphed”, or a “rule which can be expressed algebraically or graphically” (Arnold, 1992d).

The SOLO taxonomy distinguishes levels of understanding in the acquisition of concepts which are relevant in this context. Students who focus quickly upon a single property or characteristic of a concept are said to be *unistructural*; those who recognise several properties as relevant, but do not link these together may be thought of as *multistructural;* those who are able to see the relationships between the various properties or representations of a concept are said to be *relational*. Some may go beyond this level, forming new connections and seeing applications of the concept in new situations; such learners are said to be operating at an *extended abstract* level. These levels (together with an initial *pre-structural* level) are considered to cycle through each of the developmental modes - sensori-motor, ikonic, concrete-symbolic, formal and post-formal operations. Learners who describe functions using multiple images would be considered to be operating at a higher cognitive level than those who use only a single image. Whether they see the relationships between these images (relational) or merely perceive them independently (multistructural) would be difficult to determine without individual interviews. Nonetheless, we might expect that those who describe functions using multiple representations might be more successful than those who think of them unistructurally in solving problems which require analysis of function properties. More recent developments in SOLO theory (Biggs and Collis, 1991, Collis and Biggs, 1991) suggest that individuals who operate in a *multimodal* way (able to draw upon earlier modes of thinking) will be more effective as problem solvers and critical thinkers. Students able to draw upon versatile images of function (including the idea of "function as process" and global "ikonic" images) to supplement their more usual concrete-symbolic way of thinking were found to be more capable at analysing functions and solving problems than those who tended to approach such situations unimodally (Arnold, 1992d).

The mental representations of functions have been described in a variety of ways. Eisenberg and Dreyfuss (1989) distinguish between *visual* and *symbolic* representations, similar to that which Vinner (1989) describes as *visual* and *algebraic* modes of thinking about functions.

Konvisser (1989) describes three representations - numerical, symbolic and graphical, while Tall and Thomas argue for *versatile learners* (1989) - citing the work of Brumby (1982), they describe people as global/holistic, exemplified by thinking of functions as curves, serialist/analytic, as in the "function machine" image or, more commonly, "versatile" - a mixture of the two.

A dual view of mathematical concepts such as function is further explored by Sfard (1991, 1992, 1994), who describes such concepts as having *structural* and *operational* dimensions, which should be seen as complementary, not incompatible. Viewing functions as objects (whether algebraic or graphical) is consistent with a structural conception, while the idea of function as process is an operational view. Sfard argues that the two perspectives interact and support each other, and that the operational mode precedes the structural mode in concept formation. In learning a concept such as function, students begin with an active conception (substituting numerical values into expressions) and gradually come to view such expressions as objects in their own right. Such objects may then be manipulated and analysed for their own sake, eventually becoming the basis for new processes in the formation of other higher order concepts (as in the composition of functions). This process is consistent with that described by Bruner (in Bruner and Anglin, 1973), who sees cognitive growth as moving through stages of representation which he described as *enactive*, *iconic* and *symbolic*.

The evolution of the function concept throughout secondary schooling follows this pattern. Although students begin with the enactive ideas of "function machine" or numerical substitution in junior secondary, they move quickly to the study of functions as objects, which appears to remain the focus for all future study. The valuable perception of function as process may well be lost for many students by the time they reach their senior years. Whether they are fixed at the analytic concrete-symbolic mode or the global ikonic mode, early research indicates that students unable to utilise both ways of thinking may well be disadvantaged in thinking about and using functions effectively (Arnold, 1992d). More recent work by Sfard (1992) suggests that the majority of secondary school students hold a concept of function which she terms *pseudostructural*, a superficial and inflexible understanding which results from teaching the concept as an object (that is, teaching *structurally*) before the students have established its nature and reality through exploration of the *operational* dimension (Sfard, 1992, pp. 75-77).

As the use of computer technology has begun to impact more and more upon senior mathematics classrooms in the form of *graph plotting* software, *multiple representation software* (such as *ANUgraph* and *CC3 - the Calculus Calculator*) and *computer algebra tools* (such as *Maple V*) which allow both the representation and manipulation of functions in a variety of forms, the effects of such technology upon the ways in which students visualise and use mathematical functions become critically important. The use of such technology to improve student understanding of the concepts of function and variable, and to strengthen the links between the various representations available for such concepts is likely to become a significant factor in the effective use of computer tools in mathematics learning, and provides a central focus for the present study.

Effective Teaching in the Computer Age

In order to provide initial direction regarding the implementation of the new modes of instruction which will accompany the use of computer algebra tools in mathematics classrooms, it is relevant to consider previous research on effective teaching in a more general sense. In particular, does the extensive research on teaching reveal particular instructional behaviours, teaching strategies or modes of thinking and representation which are likely to aid in the successful implementation of the current technological innovation? Studies on the classroom use of computers in a variety of contexts indicate considerable promise as demonstration tools (Ganguli, 1992), especially with regard to improved concept development, and for encouraging a more individualised instructional mode (Hativa *et al*, 1990). Work with pre-schoolers suggests that young children interact naturally and effectively with computers without the level of adult intervention previously considered necessary (Hall and Elliott, 1992). However, there is also evidence that teachers are unlikely to alter their instructional patterns to any real extent in order to incorporate classroom computer technology - even to the extent that they are unlikely to rearrange pre-existing spatial arrangements or alter the sequence or mode of instruction (Mehan, 1989). There was evidence, however, that students were more adaptable in such circumstances, and showed increased levels of mutual assistance and co-operation. The problems of incorporating computer technology into effective instruction, then, may well lie far more with teachers than with the students involved.

What constitutes "effective" or even "good" teaching, of course, remains problematic, particularly in a time of "paradigm shift" (Prawat, 1992, p. 354) from previous assumptions about teaching and learning to the constructivist stance increasingly espoused by researchers, if not by practitioners (Prawat, 1992, Richardson, 1990). As new priorities for instructional outcomes are recognised, practice previously recognised as "effective" may suddenly become inadequate. A study by Schoenfeld of a teacher in a 10th-grade geometry class demonstrated exactly this effect (Schoenfeld, 1988):

> Two pictures of the instruction and its results emerged from the study. On the one hand, almost everything that took place in the classroom went as intended - both in terms of the curriculum and in terms of the quality of the instruction. The class was well managed and well taught, and the students did well on standard performance measures. Seen from this perspective, the class was quite successful. Yet from another perspective, the class was an important and illustrative failure. There were significant ways in which . . having taken the course may have done the students as much harm as good. (p. 145)

Attempts to classify teaching behaviours as "exemplary", "expert" or even "effective" must be viewed as a consequence of the assumptions about teaching and learning which one holds. In particular, the method of assessment which is used will have considerable influence upon the conclusions drawn about a particular situation. If the outcome of teaching practice is measured only in terms of student achievement on traditional examinations, then certain strategies may be considered well-defined in contributing to such results. Correlating teaching practices with class achievement in introductory algebra, LeClerc, Bertrand and Dufour concluded in agreement with Good Biddle and Brophy (1983 in LeClerc, Bertrand and Dufour, 1986),

> that pupils learn more efficiently when their teachers first structure new

> information for them and help them to relate it to what they already know, and then monitor the performance and provide corrective feedback during recitation, drill, practice, or application activities that provide pupils with opportunities to develop mastery and use what they have learned. (p. 365)

If such advice is offered as a "recipe" for "effective teaching" then, no matter how useful it may prove within certain contexts, it appears doomed to failure. The research on teaching of the past decade has revealed the complexities of the teacher's task - that teaching is a "complex cognitive skill" (Leinhardt, 1989, p. 53) which has tended to defy experimental and correlational efforts to define and categorise it (Bertrand and LeClerc, 1985). Scriven's distinction between the "quest for knowledge" and the "improvement of practice" models for research (Scriven, 1983, p. 8) appears appropriate in this context. The moves in teacher research over the past decade towards the study of "expert/novice distinctions" (Berliner, 1986, Magliaro and Borko, 1986, Carter, Sabers, Cushing, Pinnegar and Berliner, 1987, Leinhardt, Weidman and Hammond, 1987, Strahan, 1989, Leinhardt, 1989 and Livingston and Borko, 1990), "exemplary practice" (Tobin and Fraser, 1988) and naturalistic studies of elements of the teaching process (Weade and Evertson, 1988, Mehan, 1989, Hansen, 1989) all recognise the critical value of the "wisdom of practice" (Tobin and Fraser, 1988, Leinhardt, 1992) in understanding the task of teaching.

From studies of "knowledge growth in teaching", Shulman describes teacher knowledge as consisting of distinct domains, distinguishing between areas such as "content knowledge", "pedagogical content knowledge" and "curricular knowledge" (Shulman, 1986, pp. 9-10). He describes the first as the "missing paradigm" which, at that time, had been relatively unexplored by research, the previous emphasis having been upon "how teachers manage their classrooms, organise activities, allocate time and turns, structure assignments, ascribe praise and blame, formulate the levels for their questions, plan lessons, and judge general student understanding" (Shulman, 1986, p. 8). His recommendation for greater research emphasis upon subject-matter knowledge has borne fruit, particularly in mathematics education, where several studies have focused upon teachers' knowledge and understanding of central mathematical concepts such as functions and graphing, which are relevant in the present context (Stein, Baxter and Leinhardt, 1990, Even, 1990). Such studies indicate, among other things, that poorly organised or represented knowledge leads to a narrowing of instruction in terms of providing a poor foundation for future work, an overemphasis upon non-essential aspects of the concept, and failure to capitalise upon instructional opportunities for fostering connections between concepts and representations (Stein, Baxter and Leinhardt, 1990, p. 639).

In general, research intended to make explicit the nature of "effective" teaching has tended to fall into two main areas of focus. The first concerns itself with the study of teaching *behaviours*, observing and documenting classroom practices which are associated with successful instruction, usually as judged by colleagues, supervisors or students, and measured against criteria which usually include success upon achievement-based assessment. The "exemplary teaching studies" of Tobin and associates at Curtin University in Western Australia over the past five years have tended to fall largely into this category (Tobin, 1987, Tobin and Gallagher, 1987, Tobin and Fraser, 1988, Tobin, Kahle and Fraser, 1990). Such studies focused early upon such specific factors as "wait time" between question and answer (Tobin, 1987), the role of "target students" who tend to monopolise teacher attention and reduce the cognitive demands made upon other students (Tobin and Gallagher, 1987), in addition to attempts to categorise the more general factors which influence and shape classroom practice and curriculum implementation (Tobin and Fraser, 1988). Such features as classroom management, the assessment system, the use of textbooks, in addition to time demands placed upon teachers to "cover the work" at the expense of student understanding or success - all were found to act significantly to reduce the cognitive demands of the classroom activities, and the effectiveness of instruction.

Later studies focused more directly upon identifying the practices of teachers recognised by supervisors and colleagues as "effective".

> The exemplary teachers had well-managed classes and were able to concentrate on establishing a productive learning environment. Each teacher viewed teaching in terms of facilitating student learning . . Each teacher had a stated belief that students created their own knowledge as a result of active engagement in learning tasks . . . In all cases, the exemplary teachers had a thorough and comprehensive knowledge of the content they were to teach. Furthermore they had a range of teaching strategies that could be used without a great deal of conscious thought . . . teacher expectations for student performance were high, consistent and firm . . . These teachers thought and talked about teaching approaches and were receptive to ideas for change. (Tobin and Fraser, 1988, pp. 91-92)

Clearly many of the conclusions reached in this study represent a shift away from the observation of practice and point towards the other broad field of research into effective teaching - that of teacher thinking (Mitchell and Marland, 1989). Studies such as the naturalistic investigation conducted by Magliaro and Borko (1986) demonstrated that effective instruction could not be defined simply in terms of teaching behaviours and strategies, but needed to take into account the cognitive processes which accompanied instruction. This study attempted to define relationships between classroom variables (such as participation, task structures, and time engaged on reading tasks) and student achievement in reading, contrasting two student teachers with their supervisors. The results were largely inconclusive for these factors - differences in student outcome could not be explained by the variables in question, but related more to beliefs about teaching and conceptions of their role by the participants (Magliaro and Borko, 1986; 133-135).

The recognition of teaching as a “complex cognitive skill” led naturally to studies founded upon the expert/novice distinction as the basis for studying and defining the thinking which is associated with successful teaching practice. Drawing upon studies of the thinking of experts in other complex cognitive fields (such as chess playing, note taking and solving physics problems), expert/novice studies of teaching investigated such factors as “mental scaffolding” used by teachers during instruction (Peterson and Comeaux, 1987), the introduction and integration of classroom routines (Leinhardt, Weidman and Hammond, 1987), processing and using information about students (Carter, Sabers, Cushing, Pinnegar and Berliner, 1987), agendas, lesson structures and explanations in mathematics lessons (Leinhardt, 1989), views of instruction (Strahan, 1989) and the planning and implementation of review lessons in high school mathematics (Livingston and Borko, 1990).

For all their diversity of focus, these studies are surprisingly consistent in their findings. Experts and novices differ in their recall, representation and analysis of classroom situations (Leinhardt, 1989):

> Expertise is characterised by speed of action, forward-directed solutions, accuracy, enriched representations, and elaborations of knowledge rich in depth and organisational quality, [and by] lessons that are open, flexible, responsive, problem-based and intricate. (pp. 73-74)

The phenomenon of "chunking" was frequently identified in the processes of experts (Strahan, 1989, Peterson and Comeaux, 1987) by which they were able to represent complex situations in simpler, more automated forms, described by Livingston and Borko (1990, p. 373) as "rich, well-developed, interconnected and easily accessible cognitive schema". Such cognitive organisation allowed successful teachers to recall large amounts of relevant classroom information (Carter *et al*, 1987), and quickly and effectively analyse quite complex classroom situations (Peterson and Comeaux, 1987, p. 321).

In terms of the SOLO Taxonomy, such cognitive organisation is represented by the difference between *multistructural* and *relational* levels of thinking. Whereas novice teachers tend to view the classroom situation as consisting of a large number of discrete and interwoven factors, experienced teachers seem able to draw relations between these diverse elements, and so to view the same scene as simpler in structure, and yet revealing of deeper meaning. Carter *et al* (1987) noted that experts seemed little interested in remembering much specific information about students - rather they merged the available information into a "group" picture (Carter *et al*, 1987, p. 150). Strahan (1989), using semantic ordered trees, found that experienced teachers constructed more complex and intricate representations of classroom events than novices, and expressed more student-centred views of teaching (Strahan, 1989, p. 64). Experts, too, made extensive use of classroom routines as a means of reducing cognitive load, and automating procedures by which effective learning could be facilitated (Leinhardt, Weidman and Hammond, 1987).

Such studies, while rich in descriptive and explanatory power, still fail to provide an effective means for improving practice. Just as the earlier studies of successful teaching strategies and behaviours do not provide the means by which novices or unsuccessful teachers might become “more expert”, knowledge of the ways in which experts think seems unlikely to fare any better in this regard. The transition from multistructural to relational classroom thinking cannot be accomplished easily; the very nature of teaching as a “complex cognitive skill” precludes the possibility of a “quick fix”. Similarly, in the context of the present study, it seems unlikely that the introduction of an innovation such as computer algebra software will result in changes to the ways in which the teachers’ cognitively organise their classroom interactions. The relevant question in this context concerns the likely effects upon experienced teachers when placed in a “novice” situation.

The implications of the preceding research suggest that the likely effects of such a classroom change will be upon teachers *pedagogic content knowledge* rather than their *content knowledge*. In fact, it is the relationship between these two which remains unclear. One of the points of focus in the proposed study will concern the effects of severing the links between *what* the teacher knows, and the ways in which this content may best be structured and presented in order for effective learning to occur. The research literature invariably lists expertise in subject matter as a necessary prerequisite for successful teaching

(Tobin and Fraser, 1989, Stein, Baxter and Leinhardt, 1990), but the nature of the relationship between the various forms of teaching knowledge remains unspecified. Computer algebra tools may be seen as a means of making explicit some significant aspects of these linkages.

The two approaches to research on teaching which have been examined above both contain elements of a “scientific” or “analytic” paradigm, the one focusing upon teaching behaviours and strategies, the other upon cognitive schemas and thinking. A third option exists which attempts to view teaching in a more global or holistic way, seeing it as a practice which bears many of the characteristics of art, rather than science (Zahorik, 1987), and which describes the knowledge of the successful teacher in terms of “craft” or “working” knowledge which is intuitive, difficult to verbalise and complex (Leinhardt, 1990, Gersten, Woodward and Morvant, 1992). Such approaches are often framed within a constructivist perspective, which recognises the importance of prior experience, existing beliefs and multiple kinds of knowledge in the understanding of teacher practice, and how teachers learn and change these practices (Sigel, 1984; Leinhardt, 1992, Prawat, 1992). It is perhaps instructive to note that proponents of the “teacher behaviour” model such as Tobin, and “cognitive scientists” such as Leinhardt are now working with these more global approaches to the understanding of the complex act of teaching.

The work of Tobin and colleagues in recent years, in particular, has moved away from the extensive study of classroom practices and strategies and towards the investigation of a particular aspect of teacher thinking associated with images and metaphors (Tobin, 1990, Tobin, Kahle and Fraser, 1990, Ritchie and Russell, 1991). Teachers' use of metaphors has been identified as a means by which many particular beliefs and practices may be categorised and, in some instances, changed. Such an approach is offered as potentially a means by which significant change in teacher beliefs and practices may be achieved (Tobin, 1992):

> Identification of salient teaching roles, and the metaphors used to conceptualize them, offers the possibility of changing what teachers do in the classroom. The metaphor used to make sense of a role is a master switch for associated belief sets of teachers . . . Reconceptualizing a role in terms of a new metaphor appears to switch an entirely different set of beliefs into operation. (p. 6)

Teachers who are encouraged to identify and critically examine their existing metaphors for instruction (which may include such descriptors as "captain of the ship", "entertainer", "policeman", "teacher as resource", "social director" and "travel agent") may then construct new metaphors which are perceived as more consistent with a desirable change in teaching practice. Thus, in the study by Ritchie and Russell (1991) an experienced teacher identified the metaphor of "teacher as travel agent" in such a way as to be more consistent with a constructivist mode than her current practice, developed and adopted the metaphor, and subsequently facilitated a change in instructional practice in the desired direction.

Such a model, then, offers the possibility of encouraging and facilitating teacher change in a way which is intuitive and appealing to practitioners, in that it does not involve identifying and attending to an array of variables and particular practices, but occurs on a deeper, less rational level. It seems likely that such an approach moves the focus of teaching from the *concrete-symbolic* mode of the SOLO Taxonomy to the intuitive, global *ikonic* mode. The existing research has already implied the importance of ikonic thinking by teachers, noting, for example, that "experts 'see' an entire scenario or episode before they act" (Leinhardt, 1989, p. 73), and recognising that, in addition to being a "complex, cognitive skill" teaching may also be conceived in terms of "improvisational performance" (Livingston and Borko, 1990).

The analysis and development of metaphors, then, may provide effective means by which preservice teachers may be assisted to think about changes in practice which are likely to effectively incorporate computer technology. Existing images and metaphors may provide important clues as to factors which may act to both encourage and inhibit such changes.

Three
Review of the Tools

The selection and development of appropriate mathematical tools occupied a critical position, not only in the early stages, but throughout the course of the project. Although a wide and growing range of advanced mathematical software applications exists, relatively few such tools were considered appropriate for the purposes of this study. Since the research was aimed at mathematical learning situations spanning the secondary school years, the vast majority of mathematical software tools were rejected, since most were designed for senior secondary, post-secondary or even professional mathematical applications, with little consideration given to students of lesser mathematical capability and experience.

The choice of appropriate software was governed principally by three criteria:

(1) Interface

(2) Cost, and

(3) Mathematical functionality.

These criteria are listed in order of the importance accorded to them when selecting tools for use within this study. The review which follows examines examples of three major software types - algebra, graphing and number tools - from these perspectives. In this way the eventual choice of tools may be better understood, and the particular strengths and weaknesses of each appreciated.

Algebraic Tools for Teaching and Learning

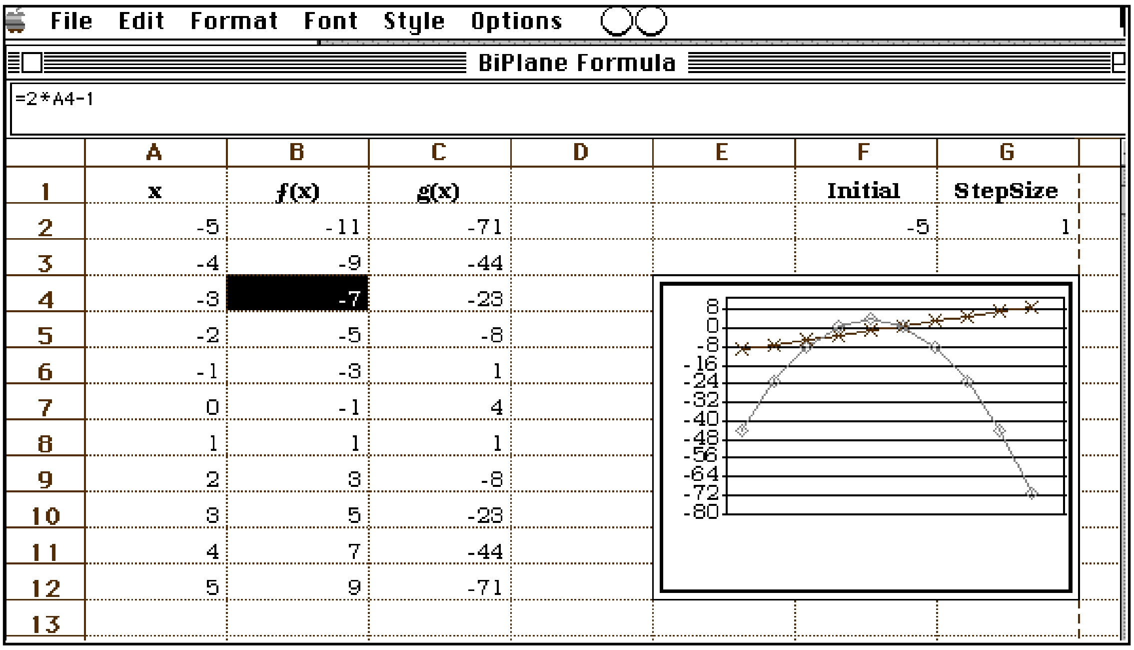
The symbolic language of mathematics was designed over time as a means of expressing the complex and powerful ideas and processes associated with the activity of *doing mathematics*, and only incidentally with the related activities of *teaching* and *learning mathematics*. Its very elegance and efficiency may in many ways serve to obscure fundamental understandings on the part of those with limited mathematical experience and, indeed, to deny entry to the “uninitiated”. When coupled with the extra demands of entering and interpreting mathematical text using a computer, additional burdens are placed upon learners who often already find difficulty enough with mathematical syntax and symbolism.

The majority of algebra and graphing software packages adopt what is, for them, the simplest approach, requiring the user to enter mathematical statements using “command-line syntax”, borrowed from computer programming. Thus, entry of an expression such as

$$\frac{1 + 2x}{\sqrt{1 - x^2}}$$

would require the user to type **(1 + 2 * x) / sqrt (1 - x^2)**. Such an entry mode not only demands that the user learns additional syntactical commands and conventions, but denies access to the important visual cues by which mathematical notation is most easily recognised and interpreted (Kirshner, 1989). While those with extensive mathematical experience may not be unduly inconvenienced by such demands, those with lesser experience, and especially those first learning the conventions of algebra, might be expected to be significantly disadvantaged.

Figure 3.1: BiPlane 2.0: A typical spreadsheet format



	A	B	C	D	E	F	G
1	x	f(x)	g(x)			Initial	StepSize
2	-5	-11	-71			-5	1
3	-4	-9	-44				
4	-3	-7	-23				
5	-2	-5	-8				
6	-1	-3	1				
7	0	-1	4				
8	1	1	1				
9	2	3	-8				
10	3	5	-23				
11	4	7	-44				
12	5	9	-71				
13							

This problem is further exacerbated in the case of spreadsheets, in which the symbolic variable is replaced by reference to a cell location, such as A3, or even A3. An algebraic formula, such as

$$3x^2 - 4x + 1$$

when adapted to a spreadsheet assumes a form such as

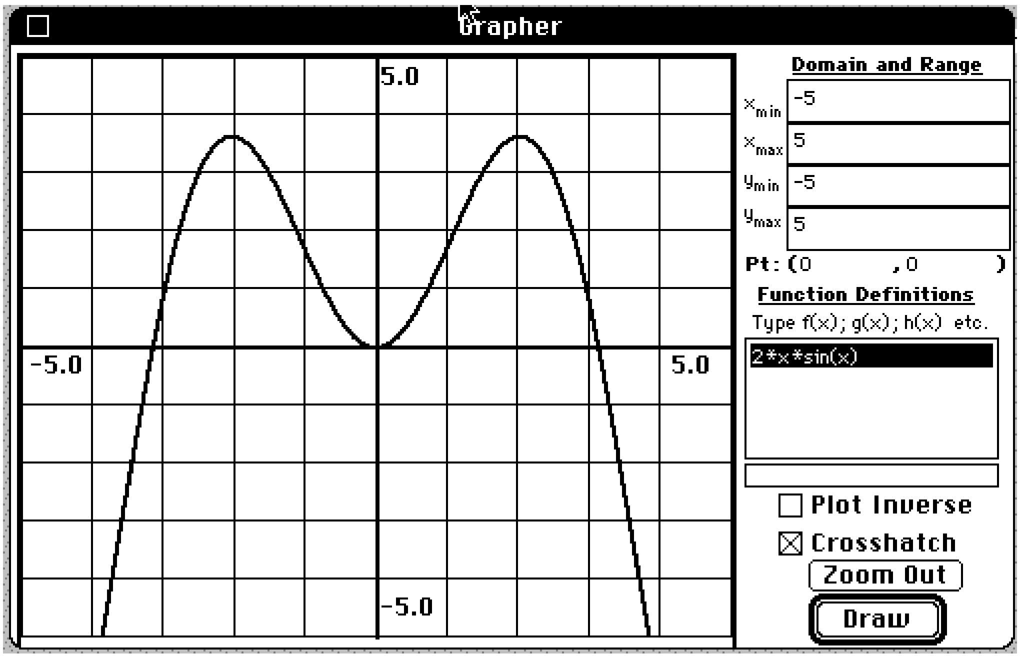
=3*A3^2-4*A3 + 1

in which the leading "=" sign indicates the commencement of a formula, and variable references point to an entry in a particular cell, A3 (see Figure 3.1). Asp, Dowsey and Stacey note the potential confusion for some students in the use of a symbol ("=") already burdened with several mathematical functions (1993b, p. 90). Some spreadsheets offer a degree of relief for this problem by allowing a cell location to be entered automatically by simply clicking on the cell, using the mouse. In this way, students do not need to enter the cell location in symbolic form, but must still specify all operations required.

This version of algebraic formatting appears to serve as both strength and weakness in defining the role of the spreadsheet as a tool for algebra learning. Although the entry and formatting of algebraic statements can be difficult and confusing, the reference to individual cell locations with distinct numerical values encourages students to perceive of variables as dynamic processes (with multiple numerical values) rather than the static placeholders so often associated with the use of symbols such as “x” and “y”. The unique capacity of the spreadsheet to expose the numerical bases for algebraic forms and expressions offers some encouragement for teachers to persevere with their use in algebra learning. It seems likely, however, that a simple “table of values” utility (in which an algebraic formula is entered in the more usual form, and features such as initial value, step size and number of steps may be controlled by the user) offers many of the advantages of the spreadsheet without the more difficult interface problems. For this reason, one of the first utilities developed for this study using *HyperCard* was a *Table of Values* tool, in which it was sought to offer the advantages of the spreadsheet in a simpler format. Designed to accompany a *HyperCard*-based graph plotter (adapted from one by Dr Khoon Yoong Wong of Murdoch University), the table of values was perceived as an important representational tool which encouraged users to perceive of algebraic forms as defined in terms of numerical processes. Dynamically linked with the graph plotter, it was seen as important for students to learn to move freely among symbolic, graphical and numerical forms.

Spreadsheets hold the honour of being the oldest mathematical software tools available for microcomputers (at least in schools), and have certainly been the among the most ubiquitous (since most school computers from the earliest days provided access to integrated software packages, traditionally offering word processing, spreadsheet and database capabilities). The difficulties associated with their use, coupled with the traditional nature of school algebra (dominated by a focus upon algebraic “objects” as opposed to the “process” approach offered by these numerical tools), have served to minimise the impact of spreadsheets upon mathematics learning situations, and may even have helped to contribute to what appears from this study to be a fairly widespread view among teachers of computers as being, for the most part, incompatible with school mathematics (Messing, 1994).

Figure 3.2: Grapher 3.61s: A typical graph plotter



If spreadsheets have largely failed to excite teachers of mathematics with their potential, the same cannot be said of graph plotters. Since spreadsheets were designed as tools for business rather than learning, the first true mathematical software available for school computers consisted of tools for plotting functions. Now increasingly available and affordable in hand-held form, the evidence of this study indicates that teachers of mathematics appear very comfortable with this application of computer technology. Participating students and student teachers had little difficulty in using graphing tools, supporting the view that these tend to sit comfortably “alongside” existing practice, as opposed to applications such as spreadsheets and computer algebra tools, which appear to critically confront such practice. Indeed, as discussed in the study which follows, while physical factors such as interface, cost and capabilities may at first appear to be the principal stumbling blocks for the use of mathematical software in schools, it is likely that there are deeper political aspects related to mathematical tool use which provide a far more influential barrier to their implementation in schools.

As specialist tools for mathematics teaching and learning have been developed over the past decade, increasingly attempts have been made to adapt these to school situations. The problem of interface has been confronted by different applications in a variety of ways. While most retain the one-dimensional format of BASIC programming, some have developed a two-dimensional approach, in which exponents are raised and multiplication is implicit. Such programs include free and shareware algebra tools for the *Macintosh* platform such as *Mathmaster 2.21* and *CoCoA 1.0c*, which both support the entry of numerical exponents using the option key, a feature adopted in the *HyperCard*-based *MathPalette*, developed for this project. This simplified entry process makes such tools readily accessible for younger students, and provides many of the important visual cues which aid in interpreting mathematical statements.

Extending this two-dimensional format are programs such as *ANUGraph*, *Milo™ 1.00* and *Theorist* (all on the *Macintosh*). These programs support the entry of mathematical expressions using menus, templates and palettes, from which mathematical forms may be chosen without the need to learn additional commands or syntactical conventions peculiar to the computer. The palette which became the basis for the *MathPalette* was based upon these principles, allowing entry entirely from visual cues and immediately producing full two-dimensional mathematical formatting. Coupled with simplified keyboard entry (such as the use of the option key for exponents and subscripts, and the “up” and “down” arrows for numerators and denominators of fractions), the palette allows quick and easy entry for both experienced and inexperienced users. Further, by converting the mathematical expression into “text-file” format at the same time, the *MathPalette* allows the user to access other software tools, within which the expression may be “pasted” using the usual *Macintosh* commands. Thus, the *MathPalette* was designed to serve as a common “front-end” for a range of available tools, and to further encourage exploration and the use of multiple representations.

The mathematical tools now available to teachers vary widely in their capabilities, their formats and, most of all, in their costs. The most expensive program considered (*Theorist* from Prescience) costs over $500 (although the Student Edition, selected for use as a principal tool in this project, costs only $105); the least expensive (*MathMaster* and

CC3) are free to be copied and distributed, with a nominal fee requested if the application is to be retained and used.

The most functionally extensive of current mathematical software, *Mathematica* (Wolfram Research), was rejected early. This was not only because it was too expensive (available at between $200 and $300 for the Student version) but also because it will not run on the type of hardware schools are likely to have available. Requiring over 6 megabytes of RAM to operate effectively, its demands are too great for the models most likely to be found in even better equipped school computer laboratories.

Similarly, *Maple V* (Brooks Cole Publishing) was considered and then rejected. Although capable of running (slowly) on the minimal machines likely to be available to schools, and offered in an affordable Student Version (at $150), the interface of this program, similar to that of *Mathematica*, was considered too difficult to support its use across the secondary years. Both packages offer extensive arrays of mathematical commands (*Maple* offers over 1400 commands), but expressions must be entered in one-dimensional form, and specific commands and syntactic forms are required. It was felt that such programs add significantly to the burden faced by students in learning algebra.

The various capabilities of these programs have been summarised in a table (Table 3.1). Applications from only two computing platforms - Macintosh and MS-DOS were considered for the project, since these offered the greatest range of possible software and appeared most prevalent in schools.

Software Summary

	Theorist	Derive	Calculus T/L II	Math-Master	xFunctions 2.2	Math-Palette
Platform	Macintosh	DOS	Macintosh	Macintosh	Macintosh	Macintosh
Cost	$105	$200	$105	Free	Free	$25
Algebra :						
Simplify	✓	✓	✓	✓	-	•
Factorise	✓	✓	✓	-	-	-
Solve equations	✓	✓	✓	•	-	✓
Substitute	✓	✓	✓	✓	✓	✓
Graph						
2 dimensional	✓	✓	✓	•	✓	✓
3 dimensional	✓	✓	✓	-	✓	-
Polar	✓	✓	✓	-	-	-
Parametric	✓	✓	✓	-	✓	✓
Calculus						
Differentiate	✓	✓	✓	-	✓	✓
Integrate (def.)	✓	✓	✓	-	-	-
Integrate (indef)	✓	✓	✓	-	✓	✓
Presentation:						
Text capabilities	✓	-	✓	-	-	•
2D notation	✓	•	✓	✓	-	✓
Other Options:						
Table of Values	✓	•	✓	-	✓	✓
Exact Arithmetic	✓	✓	✓	-	-	•
Complex Arithmetic	✓	✓	✓	-	-	•
Inequalities	•	•	•	✓	-	✓
Matrices	✓	✓	✓	-	-	-
Statistics	✓	✓	✓	-	-	-

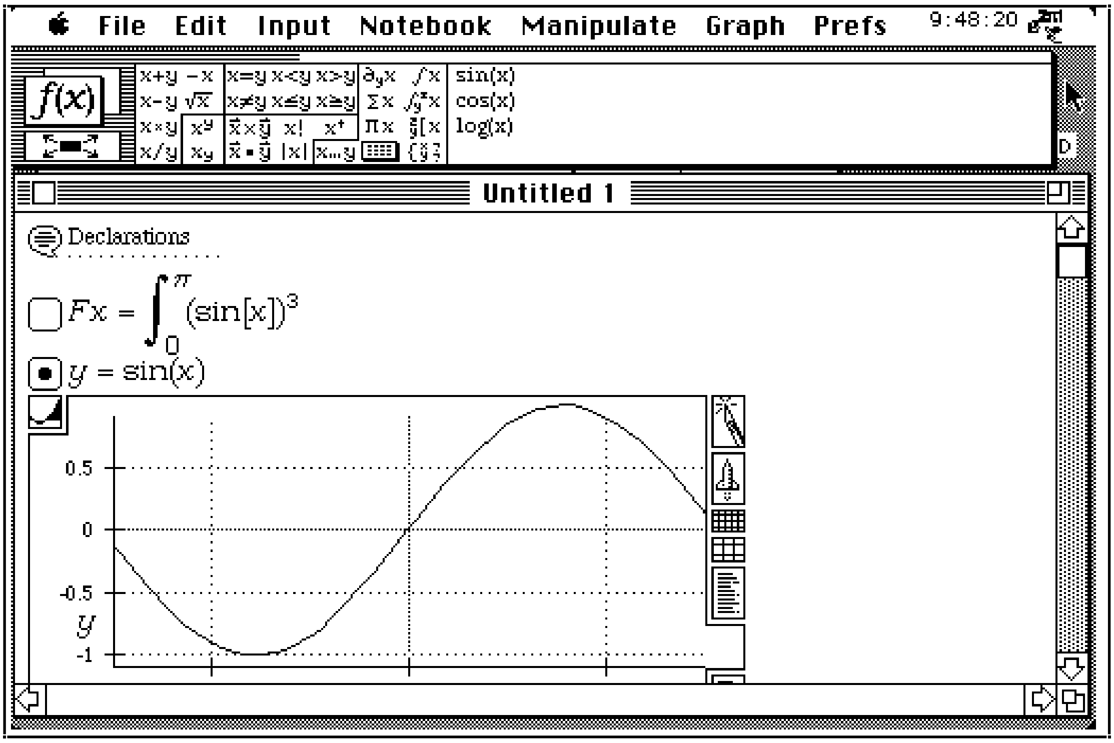
• This indicates that this feature is present in a limited way.

Selected Algebra Tools

Theorist: Almost unique in this field, *Theorist* is a Macintosh program which fully utilises the graphical interface to provide true mathematical notation in a reasonably transparent format. Almost as powerful and extensive in its rule base as *Maple*, this program appears to be a most appropriate package for teaching and learning in secondary schools. Fully menu-driven, with extensive calculus and algebraic capabilities, as well as two and three dimensional graphing and animation, mathematical expressions and equations may be entered by simply pointing and clicking at an available palette, relieving students of the need to learn additional syntactical conventions and commands in order to enter mathematical forms. Files, called “notebooks”, offer a mixture of text, graphics and mathematical forms, allowing the creation of interactive worksheets and exercises by the teacher, and annotated responses and solutions by students.

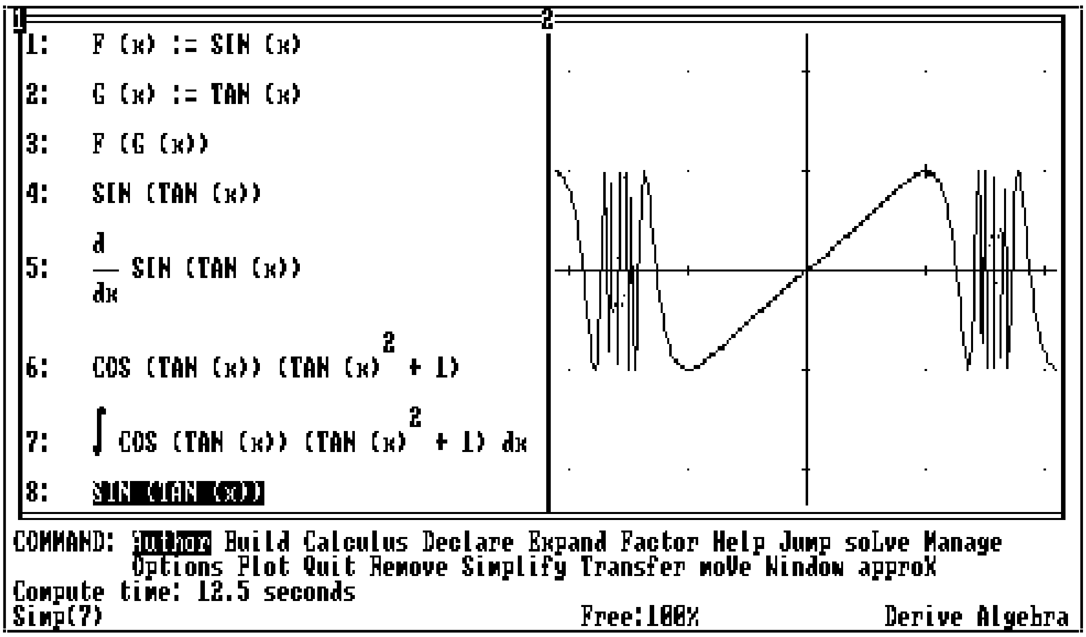
The ability of this program to manipulate terms, solve equations and substitute values into expressions using the graphical interface of the Macintosh is quite unique. Algebraic terms may be relocated by simply “dragging” with the mouse, allowing, for example, equation-solving which physically emulates methods commonly used which involve transferring terms across the equality. Graphs in both two and three dimensions (including relations such as conic sections) are simple to create and edit, and may also be manipulated by hand (dragged and rotated); these may also be animated with ease to produce a moving picture which can be used to illustrate the effects of variable changes. This program offers unprecedented control by the user over the various mathematical representations available - symbolic, graphical and tabular. It was adopted as the preferred tool for the current project. Three copies of the Student Edition were donated for the purposes of this study by the publishers (Thomas Nelson Australia), allowing it to be made available to those cooperating in the gathering of data.

Figure 3.3: *Theorist*



Derive: The MS-DOS equivalent to *Theorist* appears to be *Derive*, the successor to *muMATH* (which was the first serious attempt at computer algebra for personal computers). *Derive* is fully menu-driven, extensive in its mathematical capabilities, and presents mathematical output correctly. Input, however, must be entered in “linear” format, with the advantage of implicit multiplication (enter only 2x - 3, not 2*x - 3 as you need to do with *Maple V* and several others), and simplified (ALT key) commands for √, π, *e* and *i*.

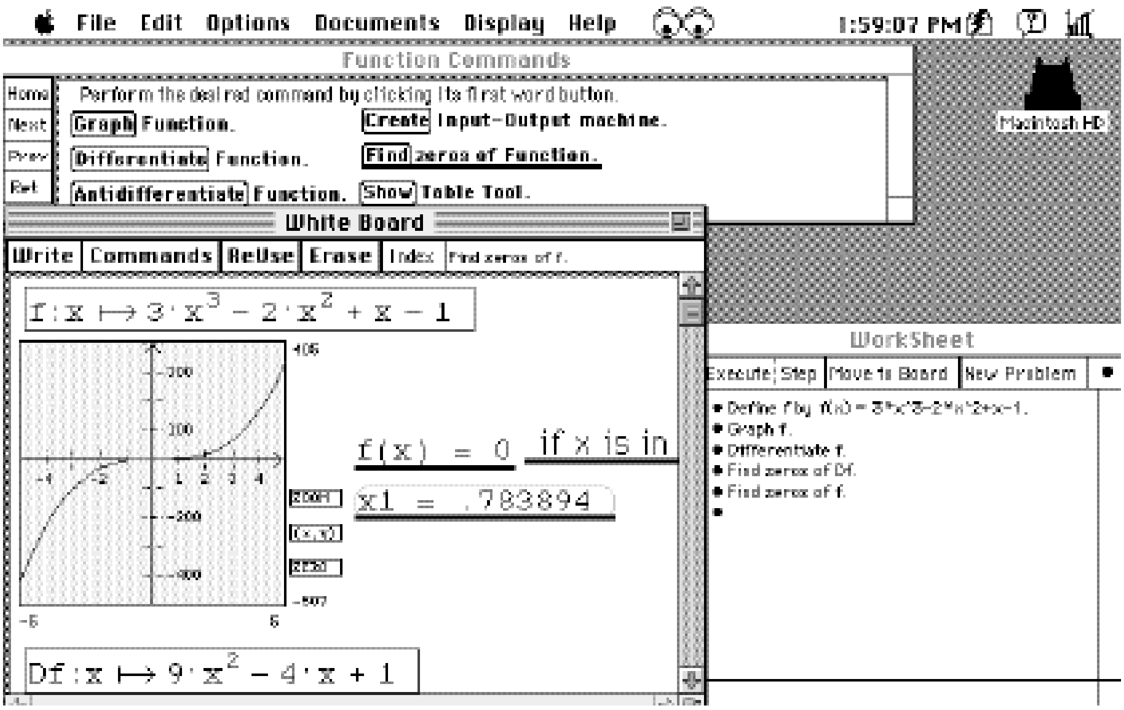
Figure 3.4: *Derive*



While not possessing the ability to freely mix text with graphics and calculations, *Derive* does give the user the option to operate up to eight different screens at the same time. It will carry out all the mathematics required up to University level, and yet is intuitively easy to use. Although expensive for individual copies (around $200), network and Lab prices are more affordable at $1195 for 10 copies networked, and $1395 for a Lab pack. Like *Theorist* on the Macintosh, this was chosen as the recommended tool for MS-DOS users in terms of ease of use and mathematical power.

Calculus T/L II: This powerful computer algebra package offers a “point and click” interface which accesses *Maple*’s algebraic “engine”. Available only on the Macintosh platform, it includes an extensive array of tutorial files, in addition to complete algebraic capabilities and two- and three-dimensional graphing. *Calculus T/L* is structured to support the inexperienced user: selecting any object on the screen provides access to the range of operations and functions which are appropriate to that object. In this way, students confronted by a “blank page” are supported in terms of their possible options, and the uncertainties often associated with problem solving and computer use are minimised.

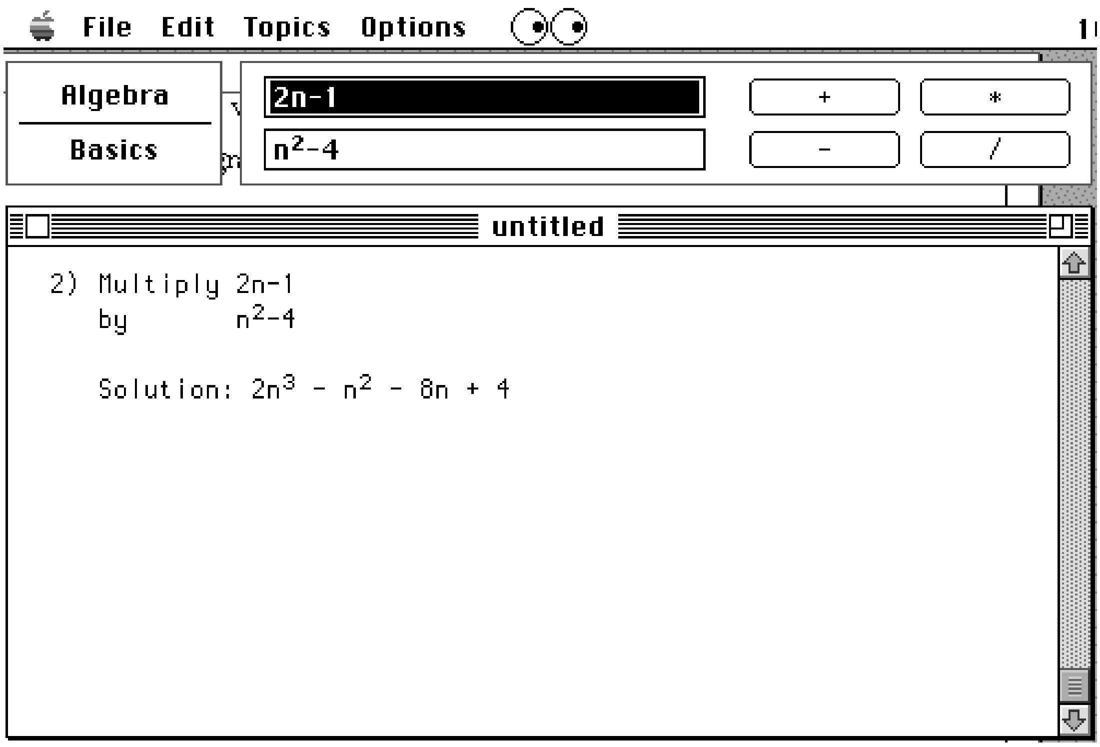
Figure 3.5: *Calculus T/L II*



Respondents using the various software tools found this to be both easy to use and extensive in its mathematical capabilities. It became the preferred option for some, even over a program such as *Theorist*, with its superior formatting and presentation.

MathMaster 2.21 : Originally released in 1987 as “shareware”, this program was withdrawn from circulation by the author, who did not wish to continue upgrading it to newer Macintosh versions and models. After some correspondence, he was persuaded to allow it to be distributed freely for educational purposes and, in particular, to be used for the current study. This is an excellent program for basic algebra and co-ordinate geometry from junior to senior years. Its capabilities include operations on numbers and polynomials (from +, -, x and ÷ to greatest common factor and lowest common denominator), simplification of algebraic expressions, solution of linear equations and inequalities (both algebraically and graphically), co-ordinate geometry (graphing and solving linear equations and inequalities in two variables, as well as finding equations given intercepts, slope, points and so on).

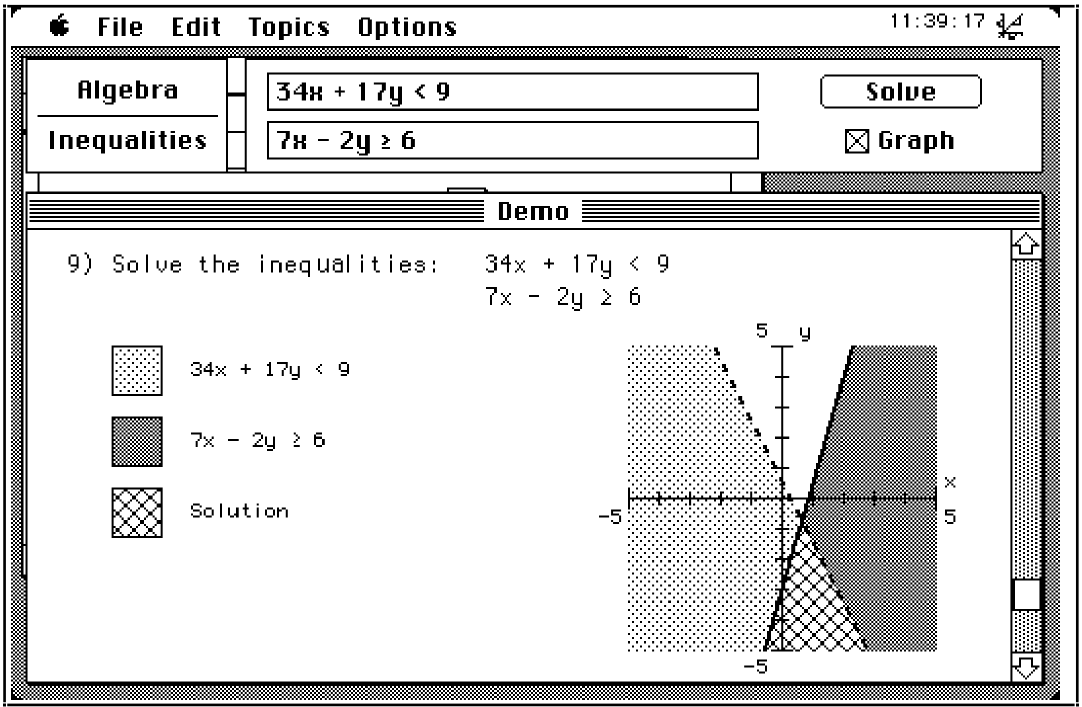
Figure 3.6: *MathMaster 2.21*



The Macintosh interface is used to advantage (including the ability to place exponents in position by using the option key with the required number). Thus an expression such as $\mathbf{3x^4y^2 - 2xy^3}$ may be entered quickly and directly. Worksheets may be created, and the results are presented with appropriate comments to describe the process involved.

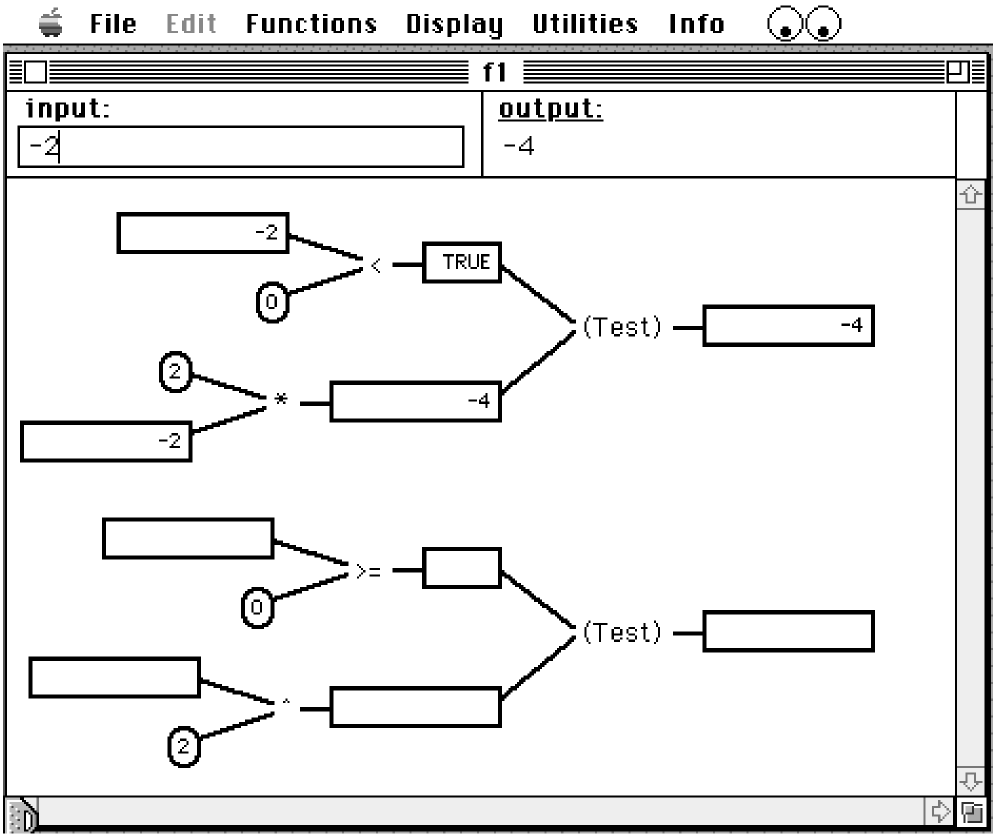
Although limited in its mathematical functionality in comparison with those programs described above, *MathMaster* offers an appropriate interface and basic algebraic capabilities which make it an attractive tool for schools which often cannot afford site license versions of the commercial programs.

Figure 3.7: *MathMaster*: Solution of Linear Inequalities



Additionally, *MathMaster* offers some particular advantages over the more powerful algebra tools. Algebraic expressions, for example, may be entered term by term, with students selecting the appropriate operations which will link them. This forces students to “reconstruct” algebraic forms from the visual printed original, which may often be perceived superficially. The coordinate geometry and equation solving features, too, provide a quick and convenient means for verifying solutions and exploring alternative forms.

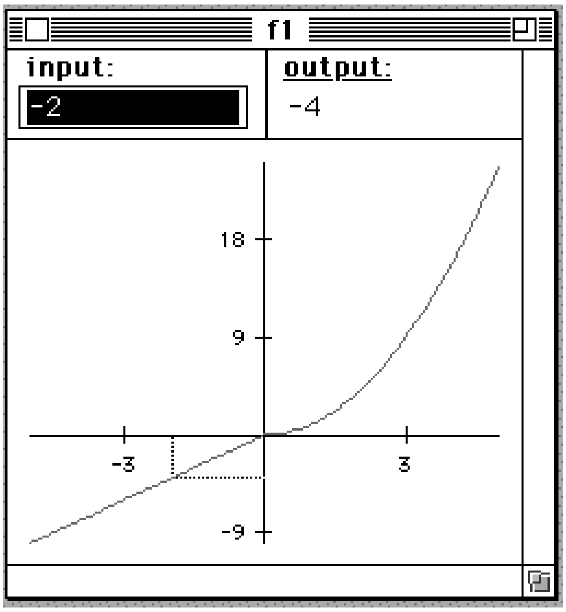
Figure 3.8: *xFunctions 2.2*



xFunctions 2.2: This multiple representational tool offers extraordinary versatility in a program freely available for educational purposes. Functions may be entered in three forms - as expressions (using the usual linear form, but allowing “split-domain” or “piece-wise” functions to be easily defined), as graphs (where the user actually creates the graph by clicking and dragging on a pair of coordinate axes), and as a table of values (entering the function values as x- and y-coordinates). Once entered, functions may be viewed in any of five representations: as expressions, graphs, tables, input-output boxes or as “diagrams”, illustrated below. Figure 3.8 displays the “diagram” for the function defined by y = 2*x for x < 0 and y = x^2 for x ≥ 0. This unique ability to expose the process of a given function appears to offer much in

assisting students to develop versatile understandings of function concepts.

Figure 3.9: *xFunctions* Graph

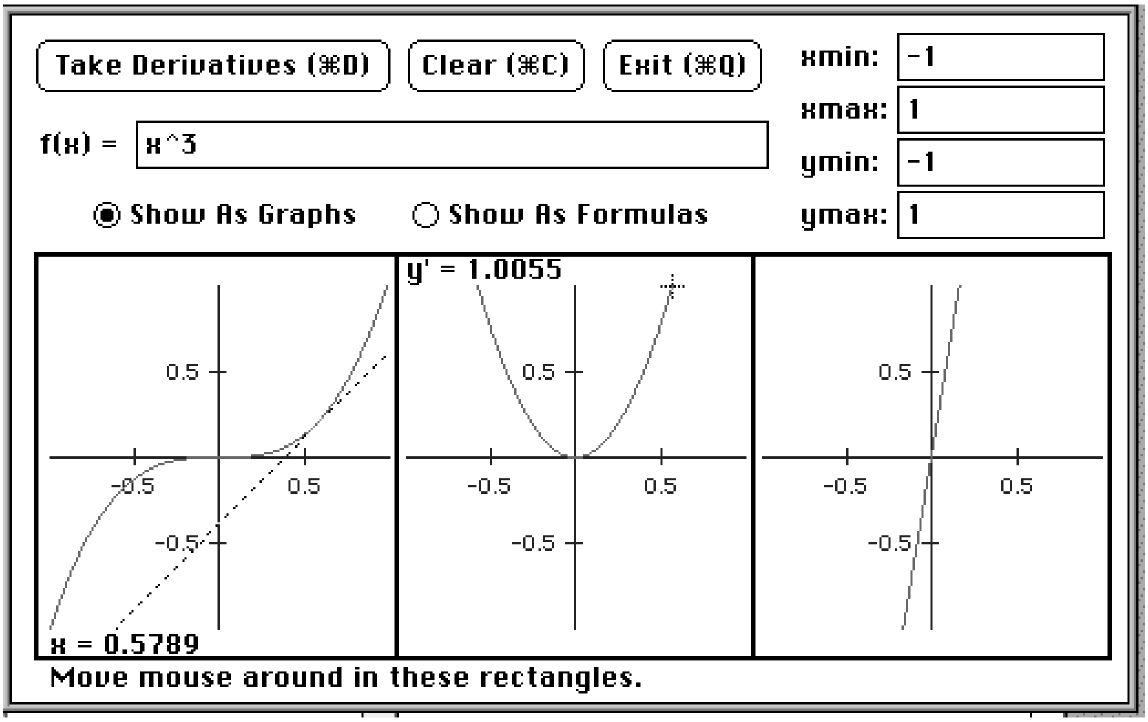


The graphing capabilities of this program are fast, clear and effective, allowing ready access to this important representation, while at the same time encouraging them to go beyond and explore other forms of the function under consideration.

In addition to the input and output capabilities of this tool, a range of mathematical features are also available, including three-dimensional plotting, animation of functions, derivatives, areas under curves, parametric forms and differential equations. This is an exceptional package for teaching and learning function concepts at all levels.

The *Derivatives* option illustrated in Figure 3.10 is typical of the capabilities of this program as a teaching and learning tool. In addition to the function, first and second derivatives are displayed, and tangent lines and points are available interactively using the mouse. Additionally, first and second derivatives are available in symbolic form, if desired.

Figure 3.10: *xFunctions* Derivatives Option



One of the few problems associated with this program is that of entry of mathematical forms. Once again, the user is forced back to one-dimensional input. It was as a particular response to this problem (and the desire to make use of the capabilities of this and other suitable programs) that consideration was given to developing a “front-end” - a program which would allow mathematical expressions to be entered easily without use of specialised syntax or commands, and then capable of accessing other software tools and “pasting” the expression in. The result was the *MathPalette*.

Conclusion

While a number of other software tools were utilised during the course of the project (including the calculator, *PCalc*, the tutorial package, *Are*

You Ready for Calculus?, the unique interactive geometry package, *Cabri-Geometrie, CoCoA* (a commutative algebra tool which offers useful facilities for expanding and simplifying polynomials, substitution and fraction capabilities) and the version of *LOGO* developed by the University of California Berkeley) the packages described above served as the principal algebraic tools for data gathering from students, teachers and student teachers over the two years of this phase of the project. When combined with the *HyperCard* modules developed for the study, participants were provided with both context and available tools for exploring algebraic ideas and skills, and the means by which such interaction might be observed. The modules, forming the package *Exploring Algebra*, are described in the following chapter on the research design. The development of the supporting multiple representational tool, *The MathPalette*, in response to interactions of participants with available tools throughout the course of the study occupies a significant part in the subsequent description of data gathering and results,

Four
The Research Design

> Classroom life, in my judgement, is too complex an affair to be viewed or talked about from any single perspective. Accordingly, as we try to grasp the meaning of what school is like for students and teachers we must not hesitate to use all the ways of knowing at our disposal. This means we must read, and look, and listen, and count things, and talk to people, and even muse introspectively over the memories of our own childhood.
>
> [Jackson, 1968 in Howe, 1988, pp. 11-12]

This study examines mathematical software use within the context of:

- characteristics of the user, in terms of both algebraic and pedagogical thinking (defined below), and
- characteristics of the learning environment, which includes the software tools themselves and the conditions under which they are used.

Within an action research framework, the study examines ways in which individuals learning about algebra make use of available mathematical software, and embodies the major findings within the ongoing development of a computer-based learning environment and accompanying mathematical tools. A theory of mathematical software use is developed using a grounded theory approach, providing a framework for future use of open-ended mathematical software tools in mathematics learning situations, and identifying features associated with both the potential and the pitfalls of the use of such tools. As explained by Strauss and Corbin (1990),

> Formulating theoretical interpretations of data grounded in reality provides a powerful means both for understanding the world 'out there' and for developing action strategies that will allow for some measure of control over it. (p. 9)

The design of the study and the subsequent gathering of data are driven by the following research questions:

- What do individuals (researcher, students and preservice teachers) understand by algebra and its components (especially functions, variables, equations, graphs and tables of values) and how might such understandings be related to the use of computer tools?
- What do individuals perceive when they view algebraic objects and how may these perceptions influence their choice and use of available strategies (including the use of mathematical software)?
- What beliefs do individuals bring with them to algebra learning situations concerning the nature of algebra, the ways in which it may best be learned, and the characteristics of successful learning and effective teaching practice? To what extent may such beliefs impact upon the use of technology as a learning strategy?
- Under what conditions do individuals choose to use available software tools, and what forms does this use take? What features of both tool and learning situation serve either to impede or encourage such use?

The Research Instrument

As a means of generating and gathering data related to the focus questions defined in Chapter One, a *HyperCard*-based research tool was developed: *Exploring Algebra* (Arnold, 1993), when used with appropriate advanced mathematical software, provides a tool by which research questions regarding the issues described above may be addressed. By providing an immediate record of user actions as they engage in the use of available tools, and making explicit aspects of their thinking and understanding arising from such use, it potentially offers a powerful means for collecting data on both pedagogical and mathematical thinking at all levels. When used in conjunction with a range of pre- and post-use data collection activities, the possibility exists for describing and explaining elements of thinking which are of critical importance in mathematics learning within a tool-based context. The research method is recursive, as the computer assumes dual roles as both object of focus and primary method of inquiry.

In order to create an appropriate research instrument, it was necessary initially to structure an “algebraic learning environment” - a series of instructional modules which attempted to synthesise the major findings of research into algebra learning of the last decade. These would then provide context for the use of additional mathematical software tools. Although the tools under investigation in this study are open-ended (presenting the user essentially with a “blank page” for computation) their use does not occur in isolation. The context of this use becomes critical in seeking to describe and understand the interactions of teachers and students with such tools. By creating and structuring such a context, the research instrument allows this variable to be controlled in the design; by recording the interactions of the users with computer tools within such contexts, it becomes possible to capture a detailed record of both action and written description of their parts. Additionally, this research and learning context itself becomes a focus for development within the action research framework of the study.

In order to better understand the design of the research tool under discussion, then, it becomes necessary to first describe its nature as an "algebraic learning environment".

The Computer as an Algebraic Learning Environment

Open-ended mathematical software of the type described here is never used in a vacuum. Rather, its use is dependent upon context and the characteristics of the individual learner. In order to focus upon these individual characteristics, and to control for the contextual element, a series of instructional modules was designed using *HyperCard* on the *Macintosh*, with the following as primary considerations:

- To provide simple and immediate access for students and teachers to the powerful computer tools by which the teaching and learning of algebra across the secondary school might be enhanced (these included computer algebra, graph plotting and tables of values as central tools, supported by *LOGO* and other quality mathematical software which the user might have available).
- To provide meaningful and challenging contexts for the use of such software tools, creating a learning environment which encourages exploration of the key concepts as they arise, and

providing the means by which such investigations might be carried out. The instructional modules developed towards this end were conceived as providing a starting point from which advanced mathematical software might be used to enhance the learning of algebra.

- To provide a means of monitoring the progress of those who would work through the program, supporting the collection of data on the path taken through the modules, the time spent at each stage, the choices made, in addition to any comments made and answers to questions in the materials. The record of transactions with the materials is collected and saved as a text file, which may then be read directly into the chosen qualitative analysis tool, *NUD•IST* (Richards and Richards, 1993).

The instructional basis for the program builds upon the considerable research of the past decade into algebra learning and misconceptions, representations and the development of concepts such as function and variable, so central to success within this domain. Such research in algebra learning appears to be converging to recommend the following changes to existing instructional and curricular patterns (Australian Education Council, 1990, NCTM, 1989, Romberg *et al*, 1993):

- A focus upon “functions” and “variables” as the organising principles for algebra learning in the secondary school, moving away from the previous focus upon “equations”.
- Increased emphasis upon teaching the ideas and manipulations of algebra linked to and embedded in real-life contexts.

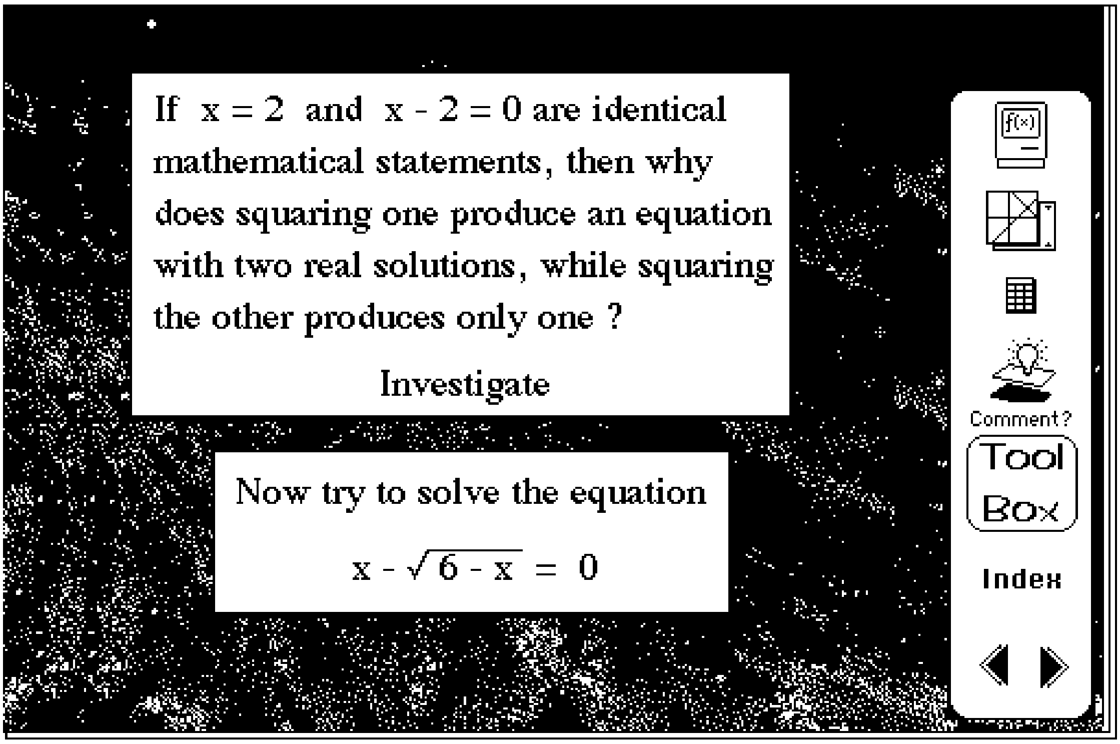
- Increased versatility in the modes of representation used to describe these algebraic ideas.
- Concrete foundations for algebraic manipulations using manipulatives (possibly linked to computer representations).
- Early and more extensive focus upon PROCESS (operational) conceptions of algebraic ideas and decreased emphasis upon the OBJECT (or structural) focus which has dominated to the present day.

The implications of such research, then, are essentially threefold. As much as possible, the algebra learning environment should provide *context* and *meaning* for the algebraic ideas and processes presented; it should increasingly emphasise the *process* dimension through frequent reference to the numerical and operational bases of the algebraic ideas they encounter, and students must experience a variety of *representations*, developing the skills to move freely among these.

As described above, the program was conceived initially as a means of placing powerful mathematical software at the fingertips of teachers and students in mathematics learning situations. As such, it was considered important to provide two scenarios - an open-ended “workspace”, where expressions of interest might be entered and investigated, supporting a range of instructional modules where examples and background information might be given, questions and problems posed, and the means and motivation provided by which the concepts of algebra may be explored.

Within the instructional modules, each screen (or “card”) provides a “control panel” down the right-hand side, with buttons which will open computer algebra, graph plotting and table of values tools. Other buttons are designed for navigation (forward and back, return to menu and index cards). A "Tool Box" button takes the user to a card from which any available software tools may be accessed; a "Comments" button allows comments, responses and criticisms to be entered at any time, to be included in the session record.

Figure 4.1: A card from *Exploring Algebra*



In order to further facilitate the use of the software tools, and to encourage exploration, mathematical expressions throughout the program made use of *hypertext* facilities: clicking on any such expression (such as the three equations shown in Figure 4.1) plots the graph; holding down the shift key produces a table of values; holding down the option key opens a selected computer algebra tool, where the expression may be entered simply by "pasting". In this way, these three powerful means of investigation and representation are automated and simplified, encouraging students and their teachers to explore the relationships between the various representations.

Exploration is also encouraged through the use of “control key” commands which are available at any time (for example, CTRL-A will open the selected computer algebra tool, CTRL-G will open a graph plotter, CTRL-S the spreadsheet) while the presence of an additional “utilities menu” at the top of each screen provides further access to the tools and other commonly used features of the software.

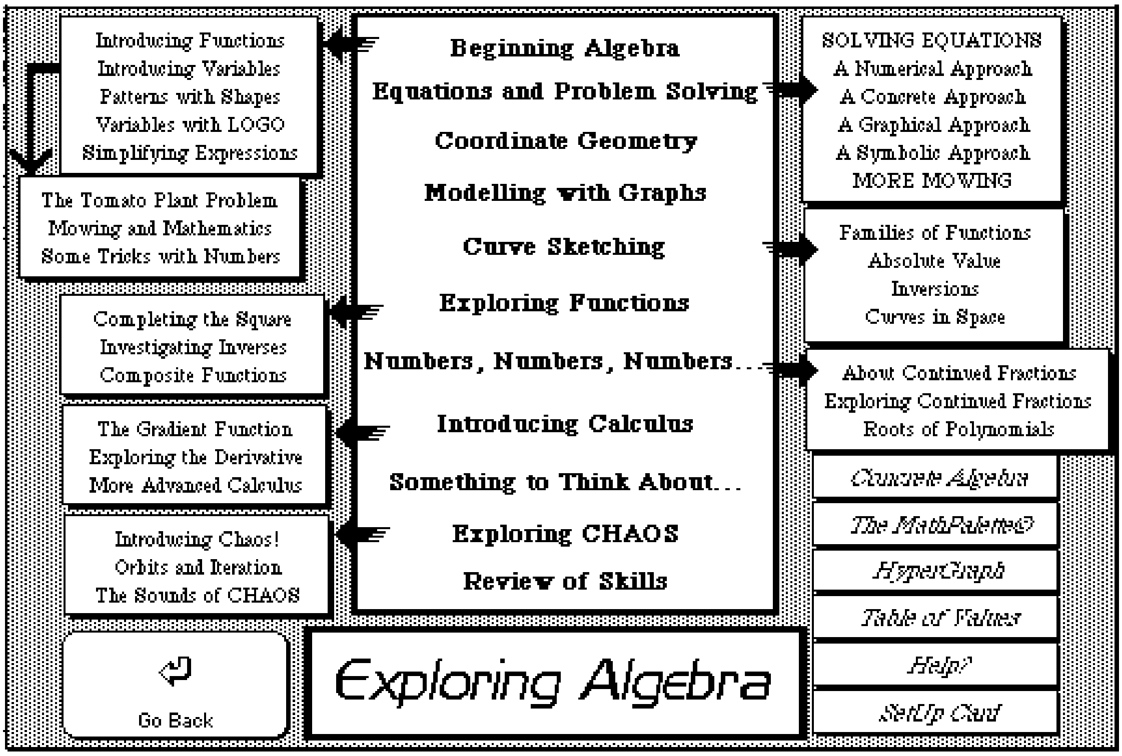
In response to evaluative comments from students and preservice teachers, the “workspace” was subsequently developed to allow the entry of algebraic expressions and equations through a “point and click” interface, removing the need for students to learn the additional syntax normally associated with algebraic and graph plotting software. Initially conceived as simply a tool for simplified entry of algebraic expressions, the development of the *MathPalette* accompanied the gathering of data from students, teachers and student teachers throughout the course of the study. As difficulties were encountered or weaknesses observed in the available software tools, so was the *MathPalette* extended and improved. The many features of this tool - the simple and mathematically correct interface, powerful graph plotting and table of values facilities, the concrete algebra modes, tools for substitution and solving, for coordinate geometry and for “Guess My Rule” games - resulted from interactions and observations over the period of the study, and reflected needs and priorities arising from the data. The research design, then, included a “research and development” component, with the latter an important key to understanding the use of software tools. The *MathPalette* accompanied the *Exploring Algebra* modules, and was available at any time within the program, encouraging student exploration and mathematical interaction.

The program, then, was designed to encourage the exploration of algebraic ideas in contexts both meaningful and versatile. Students (and their teachers) are provided with tools for developing deeper and richer understandings of the concepts of function, variable and equation, so central to success throughout the study of mathematics. As *LOGO* has been described as a mathematical “microworld”, so was *Exploring Algebra* intended to support a “learning environment” within which deeper and more versatile understandings might be possible.

The Algebraic Context and Research Questions

The mathematical context for the study was provided by the series of instructional modules developed for this purpose using the textual, graphical and interactive capabilities of *HyperCard*. Topics were chosen ranging from introductory algebraic experiences to calculus and open-ended problem-solving. These were then developed in such a way as to encourage the use of available computer tools. The materials were designed to support the development of central mathematical concepts such as function, domain and range, as well as providing context for the learning of algebraic skills, such as equation solving. An overview of the instructional modules is presented in Figure 4.2, followed by a description of each (see Arnold (1993) for a detailed transcription of the content of the modules).

Figure 4.2: Overview of the *Exploring Algebra* modules



Upon choosing a particular module, a *probe* question is asked. Four different probes exist at each section, depending upon whether the individual is a student or “teacher” (referring to preservice teachers) and whether this is the first occasion that the individual has chosen this option or a subsequent choice. The probes centre upon student thinking about the objects of algebra (functions, variables and equations) and pedagogical aspects of the module for preservice teachers. Each time a session commences, additionally, a focus probe requests an explanation of the user’s understanding of algebra, and how it is best learned. Although students were not required to answer this query every time, it provided a central recurring theme in the data from all participants. Importantly, each time an algebra, graphing or other software utility was used, a probe would follow which queried the nature of this use and the effectiveness perceived by the user. This was a critical component of the research design.

- **Beginning Algebra** provides an extensive introduction to the ideas of function and variable using a variety of approaches. Upon choosing this module, the initial teacher probe was: *"As a teacher, how would you sequence an introduction to algebra?"* (This initial probe is referred to subsequently as **T1**.)
  The subsequent teacher probe was: *"What skills and understandings do you consider essential for students to be successful in algebra?"* (Henceforth, **T2**.)
  The initial student probe was: *"What things do you consider essential for success in mathematics?"* (**S1**)
  The subsequent student probe was: *"What things do you consider essential for success in algebra?"* (**S2**)

Responses were open-ended, entered from the keyboard at the prompt. The questions were devised to deliberately explore thinking related to understanding of mathematical concepts and perceptions of effectiveness in learning. Occurring at both beginning and end of each major topic, they attempt to capture changes in thinking which may have resulted from the interaction.

The module consisted of five sections, each stressing a different aspect of introductory algebra:

(i) *Introducing functions*: a textual and graphical development of the function concept using real-world applications (families, boyfriends/girlfriends) to introduce the uniqueness property of function and, at the same time, the concept of "ordered pair" and its graphical representation. This section concluded with a mathematical application using a "function game", in which the player is invited to enter numbers and observe a numerical output, and so to guess a selection of simple "number rules". The user is *prompted* to use a simple table of values representation and, if desired, to observe the corresponding graphical form.

Probes at the conclusion of this section were:

T1: *How would you describe a function?*

T2: *How would you describe a function now?*

S1: *What does function mean to you? Please give some examples.*

S2: *Do you feel that you understand function any better now? How would you describe a function to a friend?*

(ii) *Introducing Variables*: The concept of variable is similarly introduced using real-world applications: first, the "Tomato Problem", in which students use tabular information to deduce first numerical and then symbolic relationships. This is followed by "Mowing and Mathematics", an open-ended problem-solving experience in which students are encouraged to use tables of values and graphs to initially describe and then infer advantages and disadvantages arising from the problem situation. Finally a mathematical application - "Some Tricks with Numbers", in which students play the traditional "Think of a Number" game, supported by tables of values and computer algebra. In this way participants are introduced to the three major representations and the tools by which these may be accessed.

At the end of this section, participants were again probed:

T1: *How many different types of variable could you describe?*

T2: *What does variable mean to you? Please give some examples.*

S1: *Do you feel that you understand variables any better now? How would you describe a variable to a friend?*

S2: *How would you describe a variable now?*

(iii) *Patterns with Shapes* builds further upon the use of the tabular representation as a means of thinking about situations involving variables. In this case, the derivation of numerical patterns from sequences of geometric shapes (building triangles, squares and other shapes using "matchsticks") is facilitated by a tabular array. In this form, students are invited to make conjectures and to derive both verbal and symbolic descriptions of the patterns they observe. Probe questions were:

T1: *As a teacher, how do you feel that patterns relate to functions and variables?*

T2: *As a teacher, what value do you see in using patterns to introduce algebra?*

S1: *Have these patterns helped you to understand algebra any better? If so, how?*

S2: *How do patterns relate to algebra?*

(iv) *Variables with LOGO* briefly introduces simple *LOGO* procedures for building squares, triangles and hexagons (assuming some limited prior experience) and then moves on to the use of variables as a means of generalising these figures. Students are invited to use *LOGO* and variables to further explore recursion in geometry.

(v) *Concrete materials* have increasingly assumed a significant role in early algebra, and this final section introduces formal symbolic notation using concrete representations, in which letters are assigned to represent areas of given shapes. In this way, students recognise that a letter so defined has a particular numerical value which is arbitrarily defined by the model, and that such letters may be manipulated and simplified, while retaining their numerical links. This important representation is linked to an interactive *HyperCard* model, by which students are invited to create their own algebraic expressions using shapes provided, to substitute values into these and then to evaluate the results (Figure 4.3).

Final probe questions for the module were:

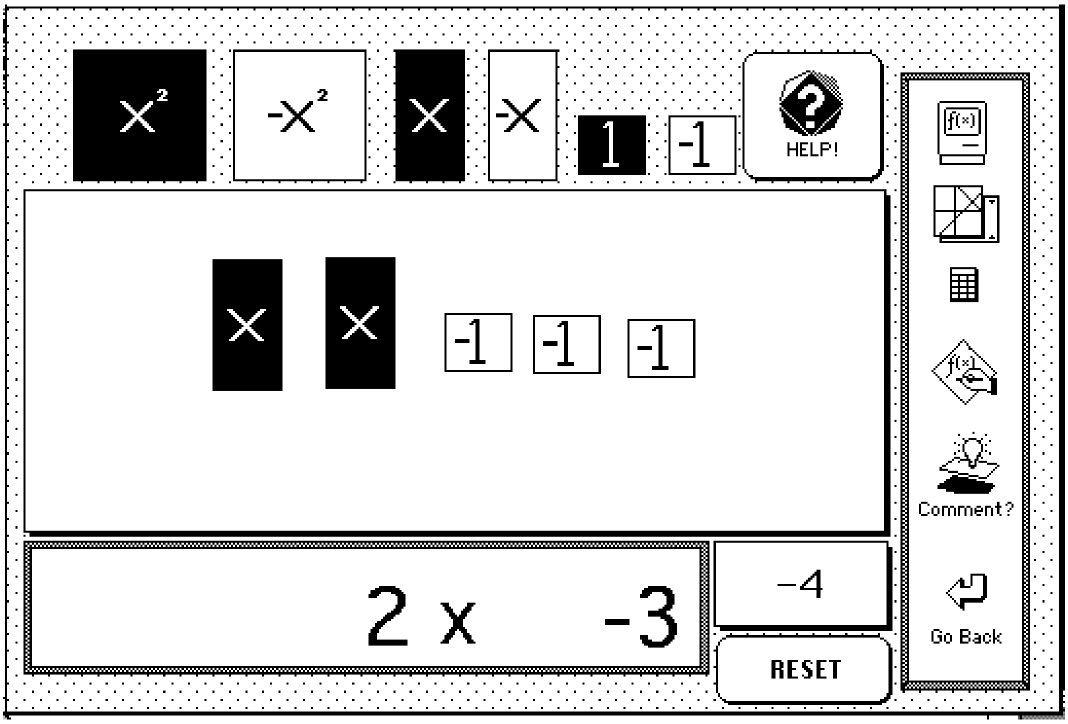
T1: *As a teacher, what value (if any) do you see in using concrete approaches in algebra?*

T2: *How might concrete approaches be used in class by students learning algebra?*

S1: *Do you feel that you understand algebra any better now? How would you explain a variable to a friend?*

S2: *Do you feel that you understand algebra any better now? Please explain.*

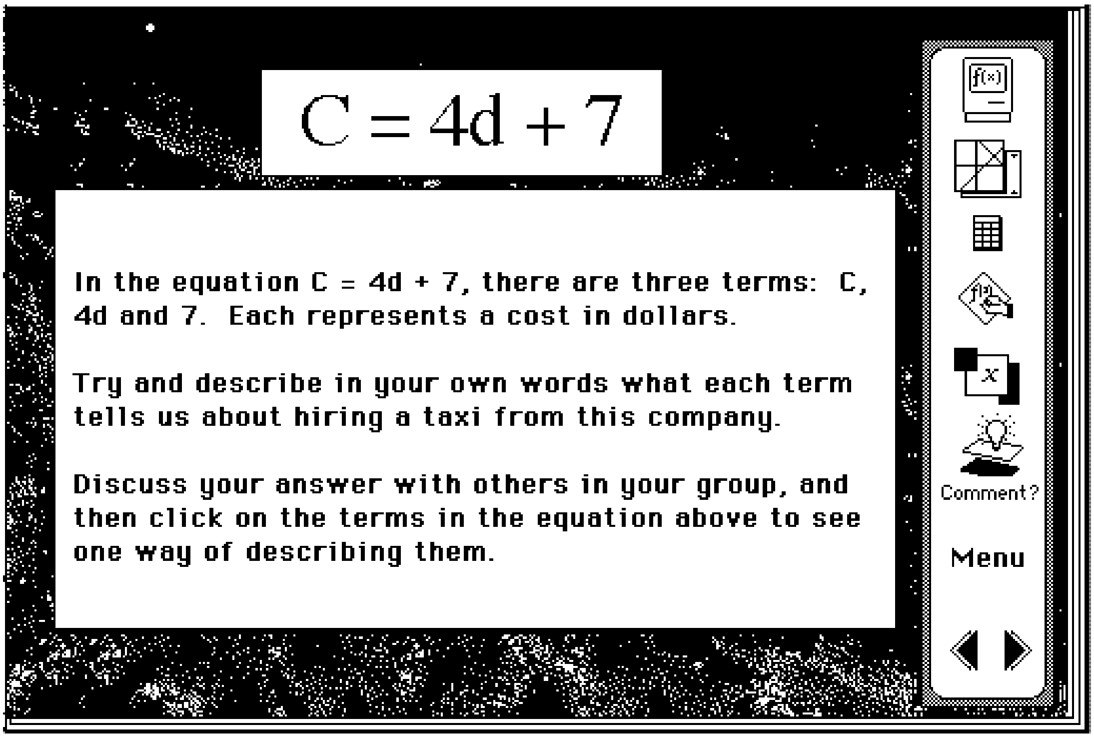
Figure 4.3: Concrete Expressions card



The module concludes with a short review section, consisting of ten multiple choice questions related to basic generalisation, simplification and substitution of values.

- **Equations and Problem Solving** continues the versatile introduction to the important ideas of function and variable applied to equation solving. Again, the manipulative aspects of the process are left until last. Emphasis is placed upon numerical, graphical and concrete methods before symbolic approaches are introduced. The module consists again of five sections, the four approaches just described, followed by an application based upon the “Mowing” problem introduced in the previous module.

Figure 4.4: Equations in context



Initial probes were:

- **T1**: *As a teacher, how would you sequence the topics in an introduction to equation solving?*
- **T2**: *What skills and understandings do you consider essential for students to be successful in solving equations?*
- **S1**: *What things do you consider essential for success in solving problems in mathematics?*
- **S2**: *What things do you consider essential for success in mathematics?*

Concluding probes were:

- **T1**: *As a teacher, how would you approach teaching equation solving now?*
- **T2**: *How might concrete approaches be used in class by students learning equation solving?*

S1: *Do you feel that you understand equations any better now? How would you describe a variable to a friend?*

S2: *Do you feel that you understand equations any better now? Please explain.*

Again, a ten-question quiz concluded the module.

- **Coordinate Geometry** is presented as a series of problems, for which students are invited to use the computer tools available. Again, manipulative aspects are deferred until students have used the various mathematical tools - midpoint, distance, gradient and equation of a line - within various contexts. These mathematical tools are readily available using either the *HyperCard* based graph plotter provided, or the mathematical application, *MathMaster 2.21*, which supports both deriving equations of lines and the solution of simultaneous linear equations and inequalities.

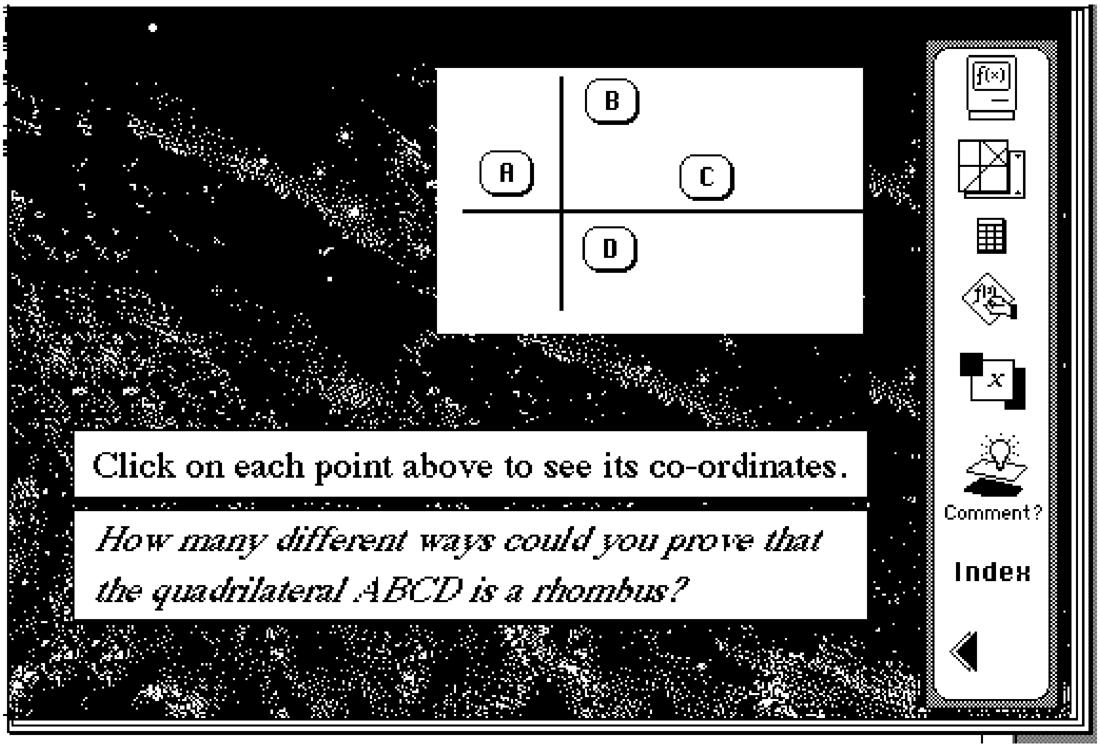
Initial probe questions were:

T1: *As a teacher, how would you sequence the topics in an introduction to coordinate geometry?*

T2: *"What skills and understandings do you consider essential for students to be successful in coordinate geometry?"*

S1/2: *What things do you consider essential for success in number plane work?*

Figure 4.5: A card from *Coordinate Geometry*



- **The Language of Graphs** takes a similar approach, providing a series of problems involving either interpretation of a given graph, or the derivation of a graph to represent a given situation. This unit was seen as an important preparation for later work involving graph interpretation, particularly in the introduction to calculus.

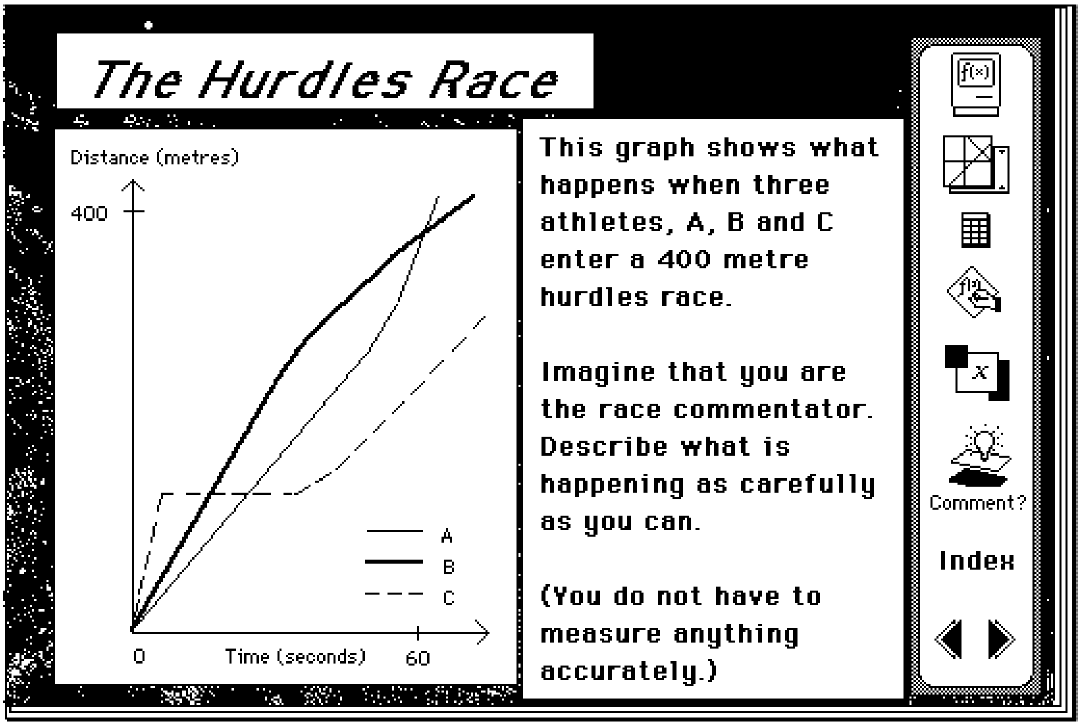
Initial probe questions for the graphs module were:

T1: *As a teacher, how would you sequence the topics in an introduction to graph interpretation?*

T2: *"What skills and understandings do you consider essential for students to be successful in interpreting graphs?"*

S1/S2: *What things do you consider essential for success in understanding graphs?*

Figure 4.6: Modelling with Graphs



- **Curve Sketching** is an extensive unit which introduces families of functions and then examines in detail absolute value and reciprocal functions, as means by which students may become familiar with both the graphical representations of algebraic forms, and the effects of transformations upon both. The module concludes with an extension involving three-dimensional graphs of functions, supported by available software tools (especially *xFunctions 2.2*).

Probes for Curve Sketching followed a similar pattern to those of previous modules:

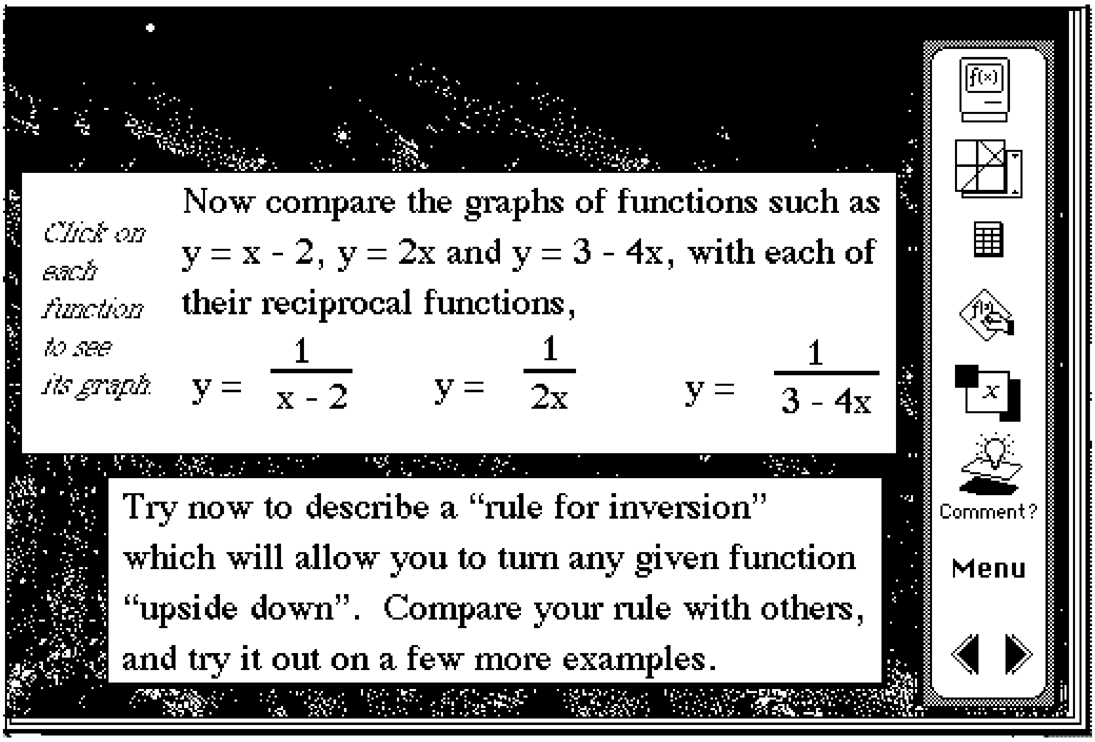
T1: *As a teacher, how would you sequence the topics in an introduction to curve sketching?*

T2: *What skills and understandings do you consider essential for students to be successful in curve sketching?*

S1: *What things do you think are most important for success in drawing graphs of functions?*

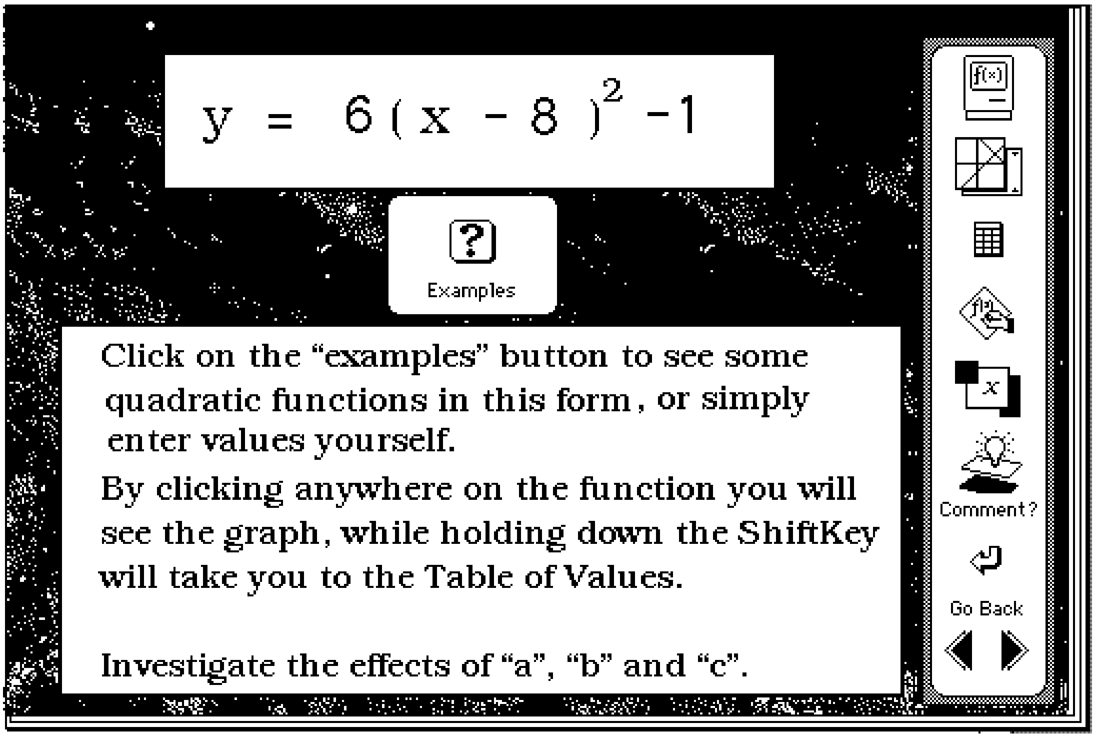
S2: *What things do you think are most important for success in curve sketching?*

Figure 4.7: Curve Sketching



- **Completing the Square** is another module which was developed extensively. It explored, not only ways in which computer algebra, graphing and tables of values might be used as tools for exploration, but also interactive aspects of the *HyperCard* programming language which supported its use as a tool for mathematical investigations. Together with the modules **Investigating Inverses** and **Composite Functions**, these provided opportunities for students to explore aspects of functions which involve higher order thinking - functions acting upon other functions to produce new functions. Each provides text-based information and interactive means of exploring the ideas presented for a variety of functions.

Figure 4.8: An application of *Completing the Square*



Initial probes for this section (entitled "Exploring Functions") were:

T1: *As a teacher, what do you see as critical to an understanding of functions?*

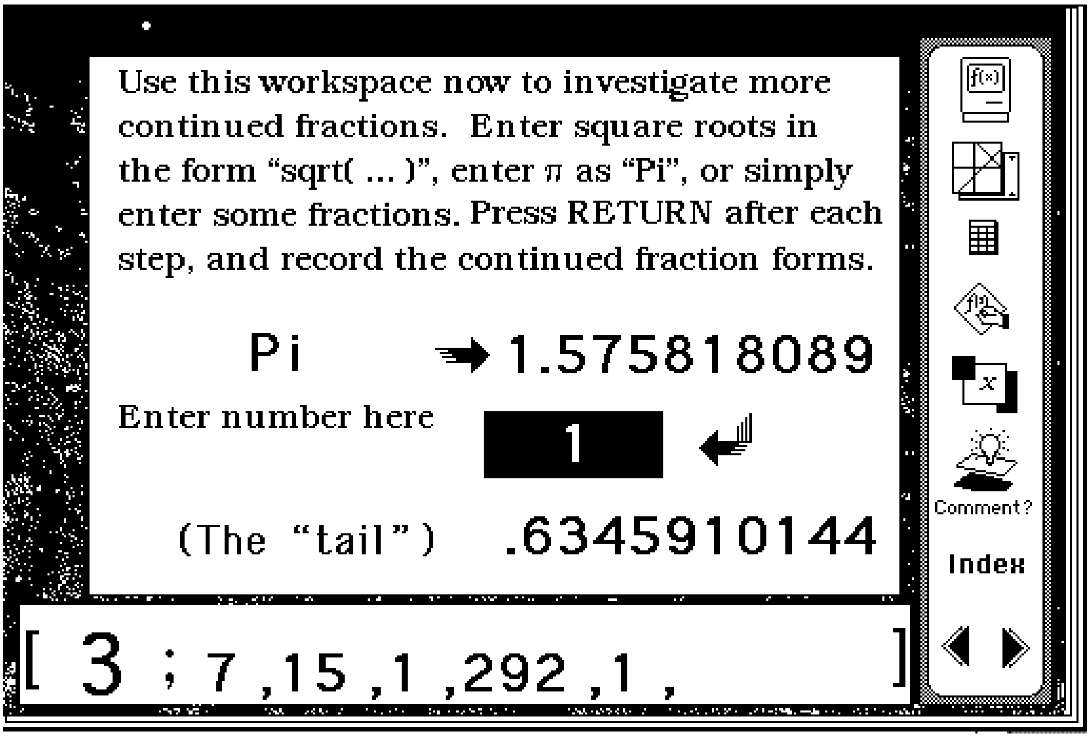
T2: *How would you explain what a function is to a student who did not know? What examples would you give?*

S1: *How would you explain what a function is to someone who did not know? What examples would you give?*

S2: *What things do you think are most important in understanding functions?*

- **Numbers, Numbers, Numbers...** offered another module which combined related units - in this case, **Continued Fractions** and **Roots of Polynomials**. Both offered means for exploring ideas related to irrational numbers, using both mathematical and computer-based tools.

Figure 4.9: A *Continued Fractions* Investigation



These units were prefaced by the following probes:

T1: *How would you sequence an introduction to the ideas of irrational numbers?*

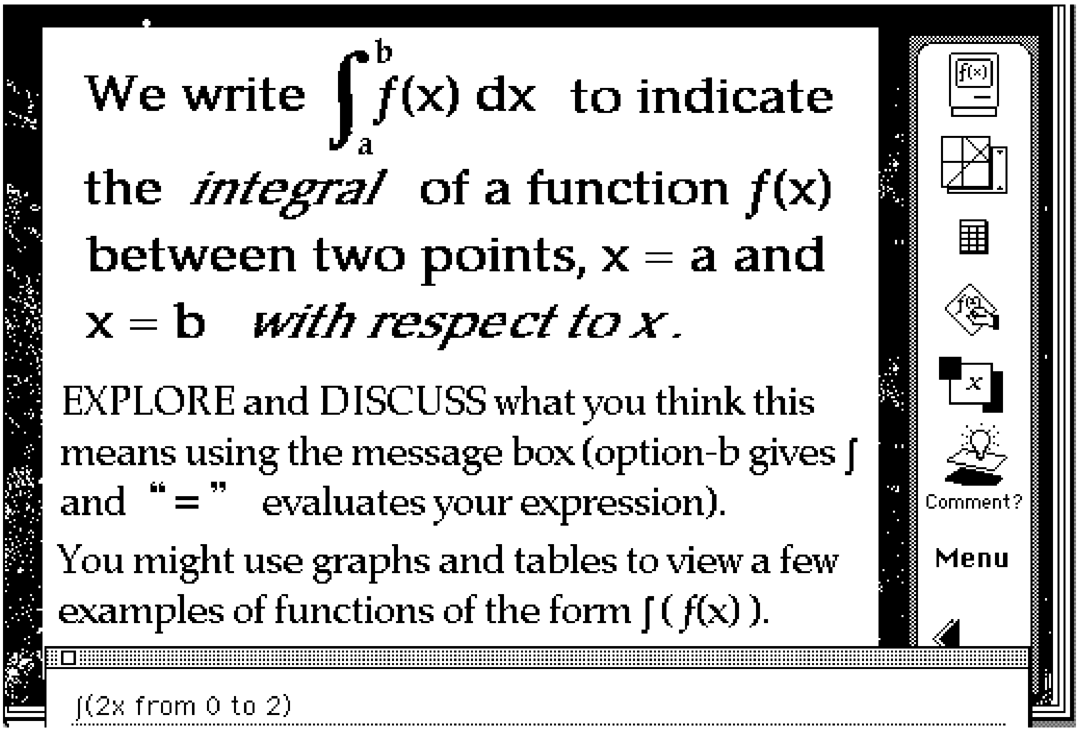
T2: *What skills and understandings do you consider essential for students to be successful in working with numbers?*

S1: *What things do you think are most important in understanding irrational numbers?*

S2: *What things do you think are most important for success in working with numbers of all types?*

- **Introduction to Calculus** was designed as a problem-based introduction, in which students could use the graphing and manipulative capabilities of the available software to “discover” the basic rules of calculus. This is in contrast to commonly-used teaching approaches in which the rules are provided by teacher or text and simply learned and practised. The module begins with ideas of “local straightness” (explored using the zooming capabilities of the technology) in order to build an understanding of the critical concept of gradient at a point.

Figure 4.10: *An Introduction to Calculus*



Again, the initial probes focus upon the teaching sequence for the student teachers and student perceptions of success.

T1: *As a teacher, how would you sequence the topics in an introduction to calculus?*

T2: *What skills and understandings do you consider essential for students to be successful in calculus?*

S1: *What things do you think are most important for success in mathematics at the higher levels?*

S2: *What things do you think are most important for success in calculus?*

This unit makes extensive use of computer support. Students are encouraged to view graphs and tables of values of the functions they encounter as a means of strengthening their familiarity with these forms. Early emphasis in the unit is placed upon the acquisition of skills of *estimation* related to derivatives - students are expected to be able to visually estimate the graph of a gradient function given the graph of the original. They are also provided with graphing and table of values facilities which represent the derivative and integral of a function without giving its symbolic form. In this way they are encouraged to discover the various rules by building upon their knowledge of functions in these representations. Finally, software tools such as *xFunctions* and *CoCoA* are available to be used to provide the symbolic derivative, if desired, allowing students facilities to verify their answers and to further explore the rules they are seeking to establish.

The module concludes with probes related to understanding of functions:

T1: *How would you describe the idea of function now? What examples might you use?*

T2: *How might computer approaches be used in class by students learning about functions?*

S1: *Do you feel that you understand functions any better now? How would you describe an equation to a friend?*

S2: *Do you feel that you understand functions any better now? Please explain.*

- **Something to Think About...** presents a collection of problems related to functions which are intended to provide the impetus for student exploration and real-world grounding and practical applications for many of the ideas and processes they have encountered. Although not all of the problems require the use of computer tools, all are inspired by the technology and encourage careful thought about the central concepts of algebra (especially function, domain and range). Some problems have been chosen to highlight the limitations as well as the advantages of computer software (for example, Figure 4.11 displays an equation for which the graphical representation provides very limited (and indeed, misleading) information, while the table of values displays the solutions immediately).

Initial probes focused again upon understanding of functions:

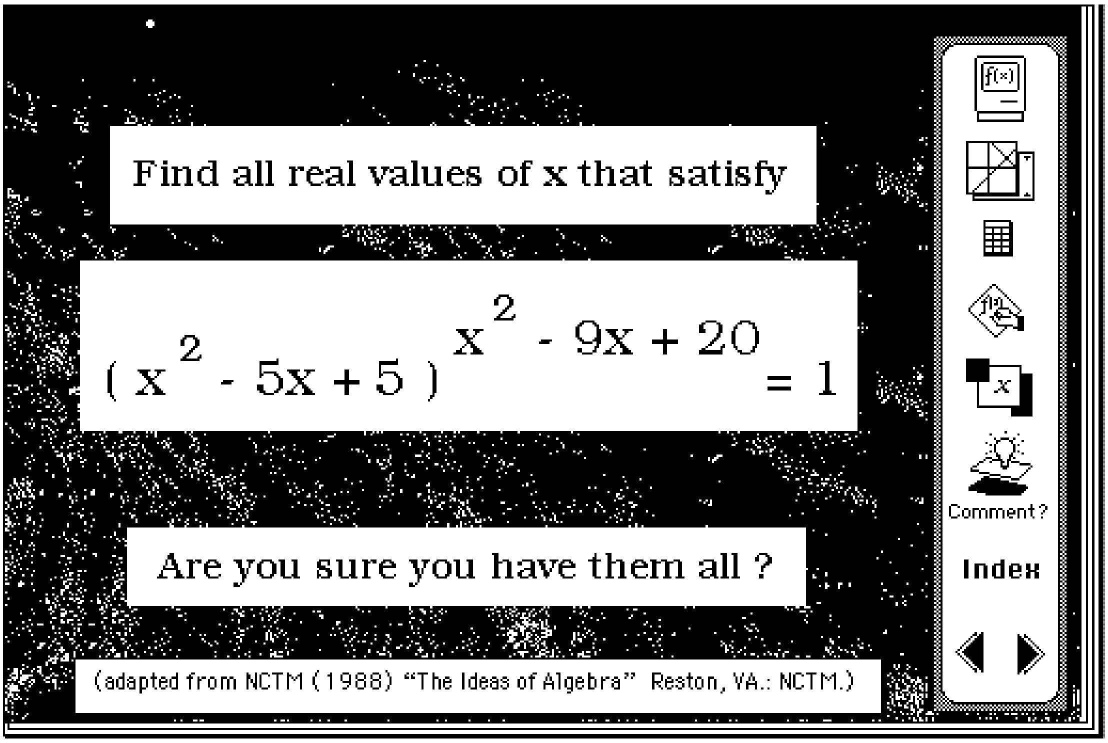
T1: *As a teacher, what aspects of functions and variables to you consider to be most important?*

T2: *As a teacher, how would you explain the idea of variable? What examples would you give?*

S1: *What things do you think are most important for success in solving problems in mathematics?*

S2: *What things do you think are most important for success in working with functions and variables?*

Figure 4.11: *Something to Think About...*



- **Exploring CHAOS** was included as an extension module which again provided impetus for mathematical investigation and open-ended problem solving. Attitudes of both students and student teachers towards such material were considered significant in the context of the use of software which demanded such exploration. Initial probes were:

T1: *As a teacher, what value (if any) do you see in introducing topics such as Chaos to students?*

T2: *What do you understand by Chaos Theory? Is it relevant to your students?*

S1: *Do you think that there is any 'new mathematics'?*

S2: *What things about Chaos do you find most interesting? What things do you think are important?*

- **Review of Skills** consisted of six ten-question quizzes which reviewed (in multiple-choice format) basic skills from Year 7 (**Beginning Algebra**), Year 8 (**Equations**), Years 9/10 (**Basic Algebra**), Year 11/12 (**Senior Algebra Review**) and challenge problems (**Stress Test**). The tests were structured in such a way that students could make several attempts, if desired. The first attempt was valued at 2 points, and no computer tools were available. This was intended to simulate the more usual mathematics learning situation, where computer tools are unavailable and students must rely upon their own mastery of algebraic skills. If their attempt was unsuccessful, the tools became available, and the value was reduced by 1. Subsequent attempts were valued at zero, placing some emphasis upon obtaining a correct answer on at least the second attempt. In this way it was anticipated that students might be encouraged to make use of the software tools to at least verify their responses on this second attempt in order to avoid a zero score. Although linked to the first two modules, the review tests could be attempted at any time and in any order.

Initial probes were included for this review:

T1: *As a teacher, what skills and understandings are essential for success in algebra?*

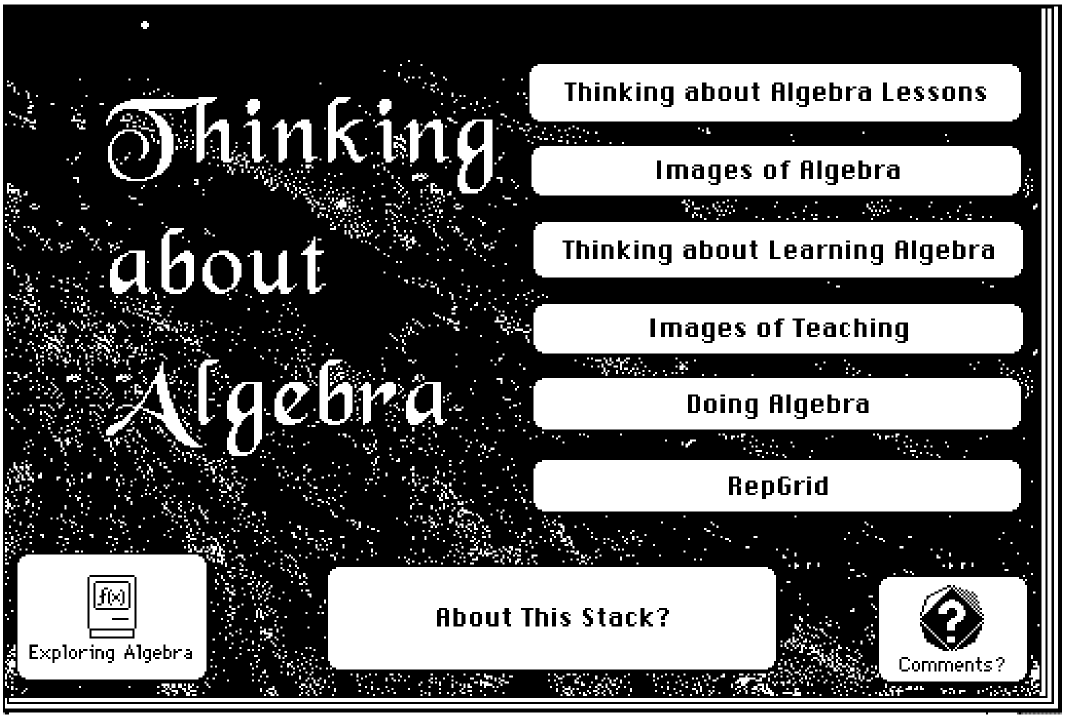
T2: *What skills and understandings do you consider essential for students to be successful in solving equations?*

S1: *What three things do you think are most important for success in mathematics?*

S2: *What three things do you think are most important for success in algebra?*

Supplementary to the algebraic learning context was a series of “research questions”, designed to provide additional data regarding the understandings, attitudes and beliefs of the participants concerning both mathematical and pedagogical aspects of algebra. At the commencement of each session, users were prompted as to whether they had “answered the research questions yet”. They were free to answer “yes” if desired and move directly into the program. If the negative response was chosen, they were presented with a menu of six parts (see Figure 4.12). Not all participants answered all parts; rather these were seen as supplementing the core data derived from the interaction of the individual with the software tools. The research questions were intended to provide a “profile” of each user which might assist in seeking to better understand their responses, and which might provide some degree of comparative data in relation to other participants.

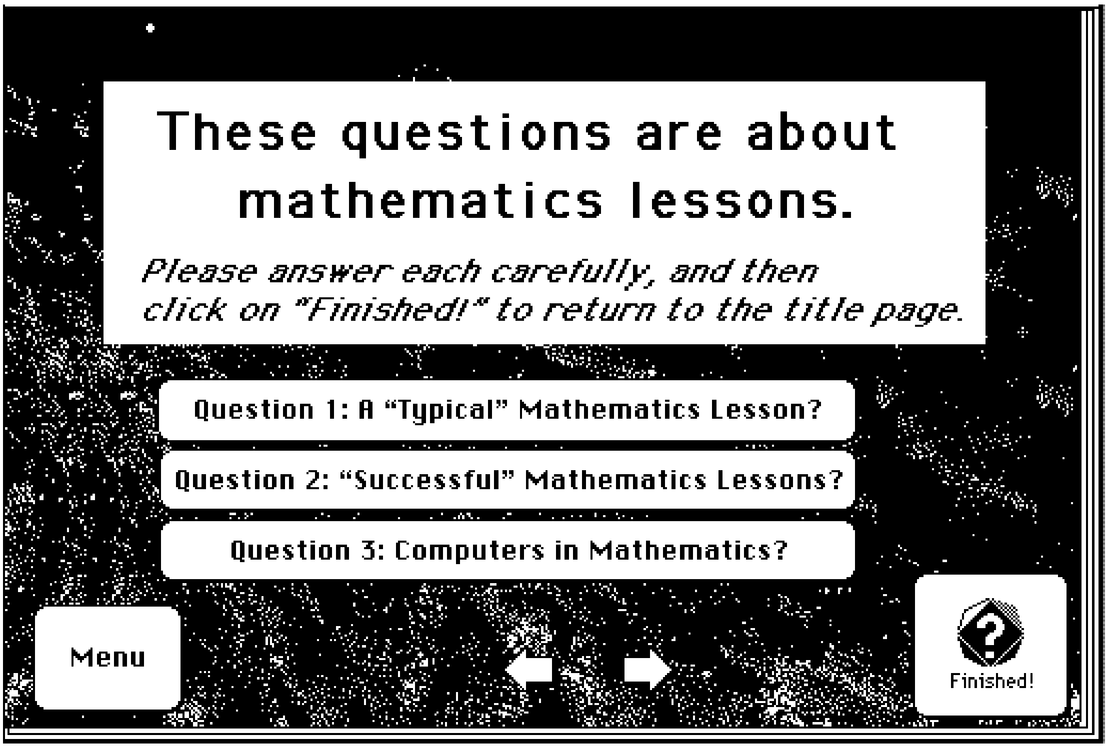
Figure 4.12: The Research Questions



Pedagogical Data about Algebra Learning

Data related to pedagogical aspects of algebra learning was generated specifically by three of the research components. The first, **Thinking about algebra lessons**, consisted of three open-ended questions, which the user was encouraged to answer "as clearly and as completely as possible". The first asked for a description of *"a 'typical' mathematics lesson"*; the second asked the participant to *"think back to a time that you experienced a really effective mathematics lesson"*. They were then required to indicate those factors which they believed helped to make the lesson so successful. Finally, the user was requested to describe *"in what ways computers might be used to make mathematics learning more effective"*.

Figure 4.13: Thinking about algebra lessons



Twin central themes recurred with regard to the data collected on pedagogical aspects of algebraic thinking throughout this study, and these are encapsulated in this first section. The notion of a "typical mathematics lesson" was considered important in helping to identify commonalities and differences across the participants with regard to their experiences of mathematics learning. Beliefs regarding the nature of a "successful" or "effective" learning experience were also considered critical in seeking to understand perceptions of the role of computers in the learning process and beliefs regarding algebra learning in general. The recurrence of these twin themes throughout the data collection process was deliberate, seeking to illuminate these concerns from different angles and so to provide some degree of triangulation from which to make judgements concerning the validity of the different responses.

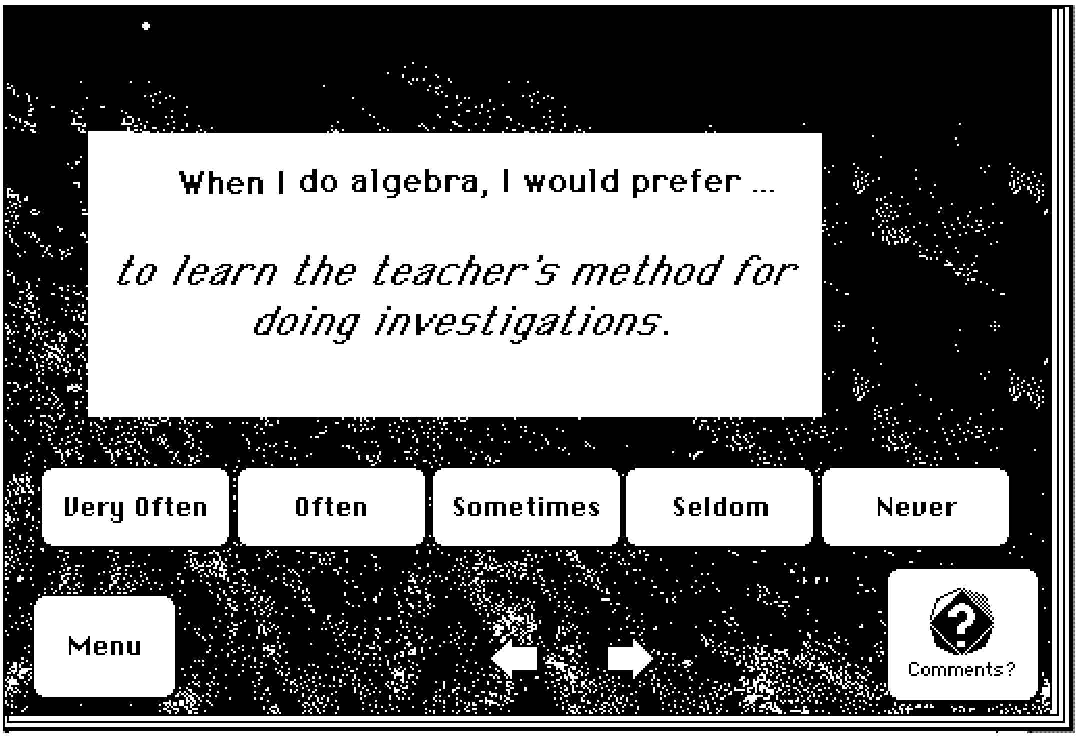
The section entitled **Thinking about learning algebra** consisted of the twenty-eight multiple-choice questions of the *Constructivist Learning Environment Scale (Preferred form)* (*CLES*) (Taylor and Fraser, 1987). Development of this scale had led to the identification of four factors associated with constructivist learning principles as defined by the authors - *autonomy, negotiation, prior knowledge* and *student-centredness*. The questionnaire adopted for this study consisted of seven questions (both positive and negative) for each of these four factors (see *Appendix B* for a listing of the items). As described by the authors (Taylor and Fraser, 1987),

> The **Autonomy** scale measures perceptions of the extent to which there are opportunities for students to exercise meaningful and deliberate control over their learning activities, and think independently of the teacher and other students. The **Prior Knowledge** scale measures perceptions of the extent to which there are opportunities for students meaningfully to integrate their prior knowledge and experiences with their newly constructed knowledge. The **Negotiation** scale measures perceptions of the extent to which there are opportunities for students to interact, negotiate meaning and build consensus. The **Student-Centredness** scale measures perceptions of the extent to which there are opportunities for students to experience learning as a process of creating and resolving personally problematic experiences. (p. 2)

The version of the *CLES scale* adapted for this study consisted of twenty-eight items in a five-point Likert format (*Very Often, Often, Sometimes, Seldom* and *Never*). The unbalanced nature of the scale responses ("very often" should be "always" in order to balance the opposite response, "never") follows the design as specified by the authors. The absolute positive response was apparently considered too extreme and unlikely to be chosen by students. Although the scale was developed in two forms - *Preferred* (in which subjects indicated responses consistent with their *preferred* mode of learning) and *Perceived* (in which responses related to the way in which they actually perceived their current mathematics learning situation), only the

Preferred version was used. While development of the scale appeared statistically rigorous, it was not intended to be used for statistical analysis in this context, but rather to provide further meaningful data for the participant profile discussed above. It was considered an appropriate instrument for this purpose since it offered quite specific and consistent information regarding participants' preferences and beliefs regarding algebra learning in a format which was well-suited to the computer-based data collection mode adopted.

Figure 4.14: *CLES* Scale item (Negative Autonomy)



The third research section directly related to pedagogical thinking was called **Images of Teaching**, and consisted of ten cards, each displaying a teaching role or metaphor, beginning with the stem “Teacher as...”.

The metaphors initially chosen were: "Teacher as..."

Entertainer

Police Officer

Gardener

Captain of the Ship

Travel Agent

Social Secretary

Tour Guide

Administrator

"The Boss"

?

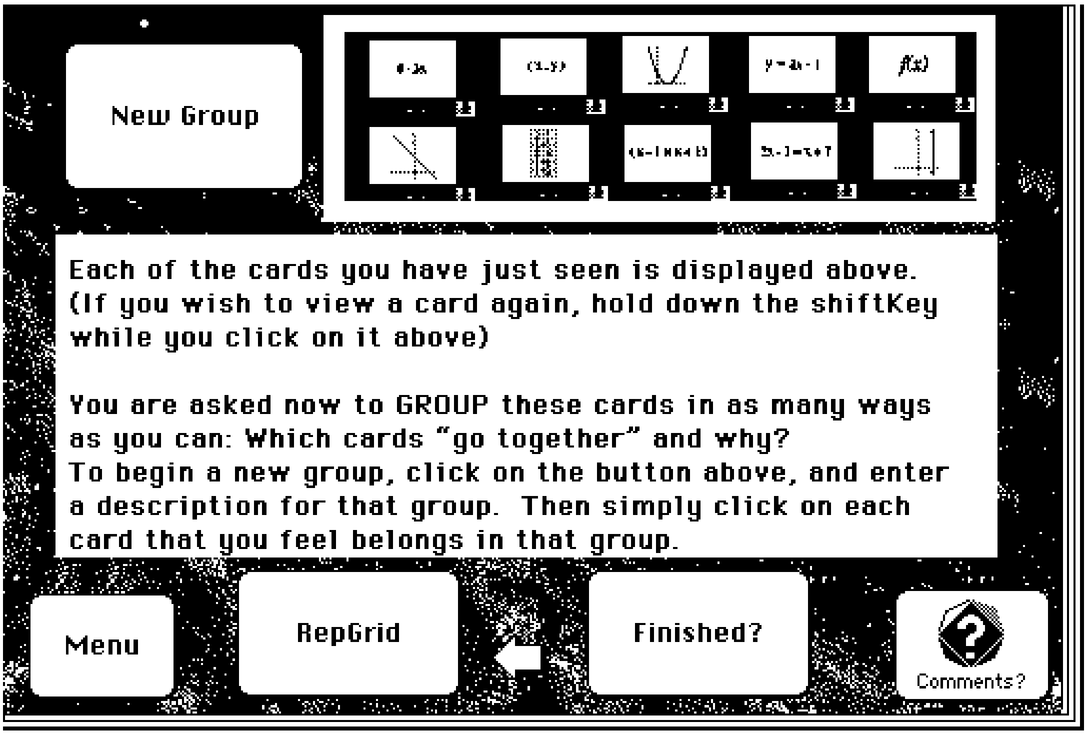
Space was provided for participants to enter one or more metaphors of their own choosing at the end. Participants were invited to describe what each metaphor meant to them and then, after viewing all cards, to associate each with more or less successful teaching. Research on the potential of metaphors as tools for understanding and improving teaching practice (Tobin, 1990, Ritchie and Russell, 1991) suggests that they might also be effective means of making explicit aspects of the craft knowledge of teaching which is otherwise difficult to articulate. For both students and preservice teachers, perceptions of the role of the teacher (especially with regard to effective lessons) would appear to be critical in the present context and provides another valuable perspective on pedagogical thinking related to algebra learning.

Mathematical Thinking about Algebra Learning

Difficulties associated with articulating tacit knowledge are not restricted to the craft knowledge of teaching. In the present context, they apply equally to thinking about algebra. Getting beyond the question "What do you think algebra is?" and, further, getting "behind" the answers given to tease out greater detail regarding both the concepts which comprise this complex notion and, more importantly, the relationships between these concepts, proved to be a critical consideration in the research design. The use of cards displaying visual images of algebraic representations as described by Stein, Baxter and Leinhardt (1990) appeared to offer an appropriate technique for eliciting deep aspects of algebraic thinking, especially within a computer-based environment. Ten images were created on cards within *HyperCard* displaying graphs, tables of values, algebraic expressions and equations and symbols related to algebra. Several were deliberately linked (e.g. the expression **$(x - 1)(x + 1)$** and a table of values displaying the rule **$y = x^2 - 1$**). Functions and non-functions were included in both symbolic and graphical forms. Within the ten cards, there was the possibility to go beyond a "surface grouping" of "graphs with graphs", "equations with equations", and so on. As with **Images of Teaching**, **Images of Algebra** invited participants to describe each card in turn, saying what each meant to them and then, after viewing them all, to *group* them in as many different ways as they could. This grouping was achieved on the computer by having the user enter a "new group" name, and then simply click on the small image of each card which belongs in that group (see Figure 4.15).

The use of visual imagery as a prompt to more detailed and rich description of complex concepts such as "algebra" appears to offer a powerful means for accessing and making explicit aspects of individual thinking. Participants at all levels asked to describe "What is algebra?" or "What does algebra mean to you?" were generally found to be extremely limited in their responses; putting complex concepts into words appears to impose quite significant demands - demands which appeared to be lessened when responses were visual and tactile, rather than linguistic.

Figure 4.15: Grouping Images of Algebra

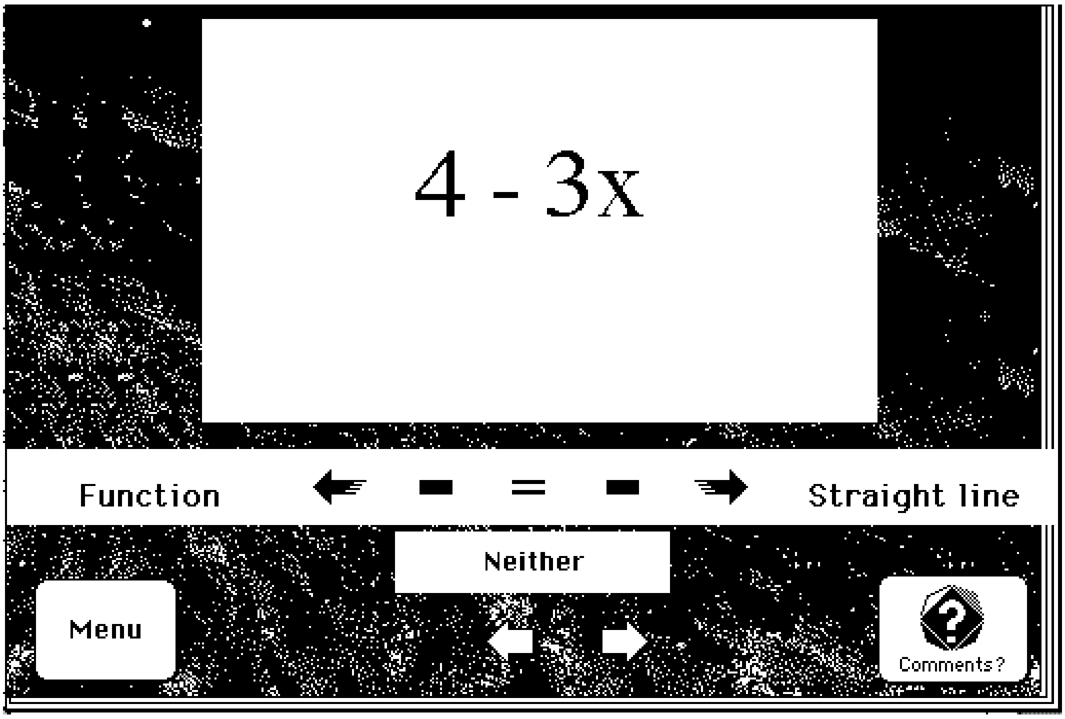


As a means of probing even further into student understanding of algebra, the **Images of Algebra** section was (for some participants) extended and enhanced using a *Repertory Grid* approach, or *RepGrid*. Based upon Kelly's (1955) Personal Construct Theory, the Repertory Grid was developed as a technique for eliciting, not just the components of individuals' thinking about complex concepts, but aspects of the relationships between these components. It has been especially popular as a tool for investigating teacher and student thinking in educational research, offering an attractive blend of data which is both detailed and idiosyncratic in its reflection of individual responses, while at the same time potentially generalisable and amenable to statistical analysis (Solas, 1993, p. 209).

A common format for *RepGrid* analysis involves deriving a series of statements or prompts related to the particular construct in question (e.g. "good teaching"), then presenting these three at a time (randomly selected) and having the participant describe "In what way are two of these alike, and different to the third". This forced discrimination generates a new series of constructs which are unique to the individual, usually in the participant's own words. Finally, these constructs may be applied back to the original prompts, where the participant commonly uses a five-point Likert format to describe the extent to which each of the constructs relates to the original statement. The resulting matrix is amenable to statistical analysis, if desired.

In this study, the ten "images of algebra" were presented three at a time, and participants were asked to indicate by clicking which was perceived to be the "odd one out", and then give a reason for this choice. For several of the student participants, the reasons were analysed to derive a number of common words or phrases (such as "equation", "function" or "graph"). These were then applied two at a time to each of the ten images as the extremes of a bipolar continuum. At each card, the participant was asked to decide whether the image shown was more like one or the other, had elements of both, or was like neither. In this way, a very detailed analysis of an individual's thinking about algebra was possible. Because the process was extremely time-consuming, only a small number of students engaged in this section.

Figure 4.16: *RepGrid* card for *Images of Algebra*



Finally, a simple attitude scale was presented (*Appendix C*). In this way, a measure of both attitude towards algebra was possible for the various participants, both at the commencement of their involvement in the project, and at various points throughout.

The Participants

This study of learning to use new tools begins as a case study of the teacher/researcher's interactions with a single student (labelled below as **S4** and referred to subsequently as *Stephen*) within a tool-rich algebraic learning context. The encounters occurred within individual tutorial situations over a period of almost two years, with some thirty-six hours of interactions recorded and analysed. As the study progressed, it grew to include five other student informants and two groups of preservice teachers as the cyclic nature of the grounded theory method demanded greater variability within the data, and new research questions and priorities became apparent.

In addition to the teacher/researcher, the study then involves three groups of participants engaged in the use of mathematical software tools. Each provides a unique perspective, intended to illuminate different aspects of the research problem. With regard to the twin poles of mathematical and pedagogical thinking, each group offers a different emphasis. The secondary students, in their dealings with the software tools, are concerned primarily with the mathematical demands of the learning contexts, and only incidentally with pedagogical aspects. The student teachers might be expected to be more interested in the pedagogical elements and implications of the experience, although one group is deliberately influenced to consider mathematical aspects. The sample groups, then, potentially offer the means of comparing and contrasting aspects of software use across different situations. Such a situation allows for considerations of variance within the study, increasing the richness of the theoretical description.

The secondary students

The student group for this study comprised six secondary students who engaged in use of a range of mathematical software within individual tutorial situations over periods ranging from two months to two years. Table 4.1 displays the characteristics of each of the individuals (including a rating of “ability level” from 1 (very high) to 5 (very low) derived from their school gradings and associated with their mathematics course for the senior students). The chosen group may be broadly viewed as providing a cross-sectional sample across gender, year level and mathematical competence. The principal informant, Stephen, is highlighted.

Table 4.1

The student participants

Code	Name	Sex	Year	Time (hours)	Ability Level
S1	Andrea	Female	11	15	2 (3 Unit)
S2	Ben	Male	12	20	3 (2 Unit)
S3	Jane	Female	10	12	3
S4	**Stephen**	**Male**	**11-12**	**36**	**2 (3 Unit)**
S5	Tony	Male	8-9	10	2
S6	Patrick	Male	7	6	2

The research sessions occurred, on average, for one hour per week during the school term. In addition to the gathering of data, the researcher made his services available as tutor for those students who desired help with their associated studies, and this aspect of the professional relationship between researcher and student was significant in influencing the nature of the data collection process. In order to create as realistic a learning situation as possible, this tutoring role was generally allowed to dominate the interactions. Activities solely related to the gathering of research data were minimised, and consequently the majority of data collected was naturalistic. These same considerations served to preclude the use of a tape recorder or video camera as means of gathering data, as such methods were considered too intrusive in what was an essentially private learning situation. Students needed to feel free to make mistakes and to display uncertainty, and such record-keeping was felt to seriously diminish such freedom.

The use of the computer as a means of recording comments and mathematical interactions proved to be an ideal compromise between the research demands and those associated with the mathematics learning situation. Students indicated a willingness to engage in this form of data collection, in full knowledge that aspects of the learning process were being recorded. Written responses to probes and prompts within the materials, and comments made at various stages in the interactions, were accepted as overt and deliberate records, which students had the option to pass over, or to answer with a degree of care and consideration denied them by other more obtrusive means.

The data collection process involving the secondary students in interactions with the various computer software tools blends elements of clinical interview procedures with participant observation. At all times, the researcher was an integral part of the process, driving and directing both the mathematics learning situation and the research component. This data collection process is considered in greater detail in subsequent sections.

The preservice teachers

Two groups of preservice teachers (hereafter described as Group A and Group B) provided data for the study. Both groups were in the final stages of four-year Bachelor of Education degrees at different institutions. They engaged in the programme as part of their assessment for units of study centred on the role of technology in the teaching of high school algebra. Group A consisted of eighteen students of whom twelve were female; Group B was made up of eight students, of whom only one was male. The strong weighting of females in both groups appears to be most closely related to the nature of the degree programme as a four-year specialist Education degree, in contrast to the other available option - the Diploma in Education, a twelve month post-graduate programme, completed after an initial generalist degree (usually Arts or Science). The latter option allows greater flexibility in career choice than the Bachelor of Education, and the gender balance in the Diploma programme was reported to be more equally distributed. It appears likely that girls were more willing than boys to make a firm commitment to the teaching profession by choosing the specialist degree course; the alternative allowed participants to “keep their options open”.

Group A: The unit studied by Group A required two hours per week class attendance over two semesters (a full academic year). The outcomes of the unit, as specified in the course outline, indicated that students would:

(i) become familiar with developments in technology appropriate for mathematics teachers.

(ii) investigate outcome based assessment.

(iii) become familiar with new changing policies in school mathematics.

(iv) compare different philosophical viewpoints of knowledge and learning.

(v) become familiar with topics in the senior New South Wales syllabus.

The coverage of the role of technology in algebra learning, then, was situated within a much broader context. Students were presented with two two-hour demonstrations of available algebra software tools (*Theorist Student Edition, MathMaster 2.21* and *xFunctions 2.2*) and were introduced to the *Exploring Algebra* package as a means by which they might investigate the use of the tools within a variety of algebra learning contexts. The remainder of their experience was then dictated by the assessment requirements for the unit.

The project on "*Algebra in Technology*" was one of three major assessment tasks specified for the unit (the other two being a Reflective Journal and a task related to outcomes based assessment), and was allocated 30% of the assessment total for the course. As described in the subject outline (Appendix D), students were required to "work through computer instructional modules and reflect on the teaching and learning of algebra in the context of computer assisted learning".

Three specialisations were made available, for junior, middle or senior algebra. The majority of students chose the junior option, probably because this was the emphasis of the demonstration of the tools. The assessment weighting favoured the pedagogic aspects of the task, with only 10% of the possible 30% allocated to the open-ended use of the software tools. This had the result that the participants focused upon issues of teaching and learning, and their exploratory use of the tools was minimal. It should be noted, too, that although *Theorist Student Edition* and *MathMaster 2.21* were demonstrated, they were unavailable to the students working though the modules due to copyright restrictions upon their use (these restrictions were later relaxed for *MathMaster* by the author, allowing it to be used freely in subsequent data collection activities).

After the class sessions introducing the software and materials, students were allocated six weeks in which to complete the assessment tasks associated with the project. They were free to access the materials on the available computers at times of their own choosing.

Group B: The second group of preservice teachers participated in a twelve hour unit specifically on the role of technology in algebra, taught by the researcher over four three-hour sessions. The course outline and assessment requirements are included in Appendix D, and deliberately favoured an exploratory approach, emphasising use of the tools within problem-based situations in addition to instructional content-based activities. Computer algebra facilities were available for this group (in the form of *MathMaster 2.21*) in addition to the modules of *Exploring Algebra* and the multiple representation tool, *xFunctions 2.2*.

References were provided to accompany each week's theme, and students were instructed to complete the assessment requirements at times convenient to them, over the duration of the unit and the four weeks following. Each of the five explorations specified in Assessment Task 1 was allocated a weighting of 12%; the other two modules were weighted at 20% each. The weighting deliberately stresses the free exploration component of the task, and encouraged students to engage actively with both the tools and the instructional materials.

As with the secondary students, ethical considerations constrained the gathering of research data in both preservice teacher groups. Once again, the priority for both groups was the learning experience, and tasks which served only a research function were largely inappropriate. A number of the students in Group A did, however, choose to work through some of the Research Questions available within the materials, providing data revealing of several aspects of their thinking about mathematical and pedagogical aspects of algebra. This group was, however, restricted in access to computer algebra software, since multiple versions of the commercial tools were not available and, at this stage, permission had not been granted to use the shareware program, *MathMaster 2.21* (such permission was subsequently granted by its author). Group B had no such software limitations and had ready access to both *MathMaster* and *CoCoA* which both offer basic algebra capabilities.

Each group, then, offers particular strengths in terms of the research data provided. Group A offers rich description of the pedagogical aspects of algebra learning, but is less strong in the mathematical aspects. Group B, on the other hand, engaged more vigorously in tool-based mathematical exploration, but offered less in the way of pedagogical data. Together, the groups complement each other well and provide detailed data regarding the ways in which these preservice teachers chose to engage in the use of mathematical software within an algebra learning context.

Analysis of Data

The varied and extensive data derived from this computer-based design were, in almost all cases, in text-file format, ready to be read directly into *NUD•IST* (Richards and Richards, 1993), the qualitative analysis software tool chosen for analysis. The use of computers as tools for qualitative data analysis is now widespread, since they provide unique and powerful means of working with relatively large amounts of textual data (and, increasingly, other formats including graphical, audio and video). *NUD•IST* is unique in providing a wide range of search and retrieval functions which may be applied, not only to the data itself (referred to in the program as the *Document System*), but also to the categories created by the researcher which are organised to form what is termed the *Index System*. The program encourages and rewards the creation of a two-dimensional tree, with each new category becoming a *node* on the tree which must be located in relation to other nodes already in existence.

This formal structure of the Index System appears highly compatible with the Grounded Theory approach, which utilises as its main tool for data analysis what is termed the *constant comparative method* (Strauss and Corbin, 1991, p. 62). As each new category is created it must be compared and contrasted with existing categories - a feature deliberately encouraged by *NUD•IST*. The Index System itself is in a constant state of flux throughout the data analysis process (which begins with the first collection of data and continues throughout the period of the project). It is intended that the Index System reflect the state of the researcher's organisation and conceptualisation of both the data and the abstract categories which arise from it.

In the present study, categorisation of the data began early in the data collection process with the identification of concepts arising directly from the data. This can rapidly lead to a proliferation of categories - over ninety categories emerged quite rapidly from the first coding process using a software package which did not support the tree-structure of *NUD•IST*. A significant advantage of this structured approach is that it actually helps to minimise the growth of new codes, since each is compared and contrasted and *situated* in relation to others.

An early form of Index System developed for this study included codes for the several theoretical positions outlined in the Introduction - Constructivism, the SOLO Taxonomy, van Hiele and Vygotskian categories. This approach was later rejected as incompatible with a Grounded Theory approach, since the theoretical structure must arise from the data, rather than being imposed upon it. This is significantly different from more traditional research approaches which are based upon theory *verification* rather than theory *generation*, as is proposed here. Alternative theoretical positions are more important at the end of the analytic process than the beginning. They serve then to support and perhaps to generalise the theoretical position which has been developed.

As these concepts are organised and refined, common themes are recognised and serve as the basis for new concepts, emerging at higher levels of abstraction and becoming increasingly *theoretical* (a process aided significantly by the keeping of journal notes and *memos* as reflective devices which assist in the movement to levels of increasing abstraction from the data). The resultant grounded theory is derived from the data by a cyclic process of *organisation* -> *abstraction* -> *theorising* -> *verification* -> *refinement*. These last stages of verification and refinement return the researcher to the data once again to reassess the categories which are contributing to the emergent theory. The data collection process itself in the later stages will also be driven by the theoretical perspective which is being developed and, increasingly, the verification stage will involve both a return to existing data and the gathering of more focused sources of information.

Strengths and Limitations of the Research Design

This study is cognitive and naturalistic in nature, seeking to elicit aspects of mathematical and pedagogical *thinking* by individuals learning algebra in a tool-based context. Since thinking itself is not open to scrutiny, it must be made explicit through consideration of the twin elements of *action* and *language*. The research instrument was designed to capture as much as possible of these elements within a particular algebra learning context. The attempt to maintain such a context in as naturalistic a mode as possible imposed certain quite significant limitations upon the design, while at the same time offering the potential for the collection of data which accurately and extensively reflects the concerns and realities for those engaged in the processes of algebra learning.

Particular limitations of the research design may be recognised as deriving from the following factors:

- *participant responses*
- *sample limitations*

- *generalisability of results*
- *researcher bias, and*
- *artificiality of the context.*

All but the last of these are common to most qualitative research studies and are frequently cited as criticisms of the Interpretivist paradigm. Any study which purports to study *thinking* must confront the most obvious hurdle - thought processes themselves are inaccessible and must be studied by inference. In particular, the assumption that language provides an adequate representation of thought may not be brushed over lightly. It was precisely this problem which Vygotsky confronted in his classic text, *Thought and Word* (1962). While denying that one is in any way an accurate mirror of the other, his fundamental thesis revolved around the links between thinking and word *meaning* (Vygotsky, 1962).

> The meaning of a word represents such a close amalgam of thought and language that it is hard to tell whether it is a phenomenon of speech or a phenomenon of thought. (p. 120)

Such has been the approach adopted in the current study. The responses elicited from the participants are seen as providing critical insights into their thinking. The more varied the response (verbal, visual, tactile) the more rigorous the connection. In the final analysis, however, we make one unavoidable assumption - that our informants are speaking the truth, providing an accurate description of their own understandings and perceptions. The relationship of the students with the researcher, built up over an extended period, coupled with the face-to-face contact of the tutorial situation, serve to alleviate the concerns for this group. The preservice teachers, however, have little or no relationship with the researcher, nor was anyone watching over their shoulders as they interacted with the software and responded to the prompts and probes - how then can their responses be trusted?

Two considerations are relevant here. The first was the nature of the research process as a significant part of their course assessment (occurring in the latter part of their studies when successful completion must hold a high priority). The second is the nature of the responses - these varied from minimal and obviously hasty answers from some, to detailed and extended tracts from others. Since the research process was a relatively time-consuming one for these students, it is assumed that those who took the time to provide lengthy and considered responses were being accurate in these. For this reason, six preservice teachers from Group A (four female, two male) who provided the most detailed responses and completed a significant part of the supplementary research questions were chosen as the principal informants. While the remaining twelve students provided useful comparative data, the emphasis in the analysis lies with the chosen six. Similarly, of the eight students in Group B, two who provided the least detail were excluded from the primary informant group.

This leads to the second recognised limitation of the study, the limitations of the sample. The absence of a third obvious group of participants - practising teachers - was a deliberate choice made late in the research process. Although their insights and perceptions would have provided valuable data, it was decided that the focus should be limited to algebra *learning* situations. If the results are to inform our understanding of the role of computers in mathematics learning and so to improve teaching practice, then we must begin by understanding how students *learn* with technology; only then can we begin to explore how teachers may best *teach* with it. The inclusion of preservice teachers was considered appropriate, since they, like high school students, are engaged in the learning of algebra, although with a different emphasis.

While the various participants chosen represent cross-sectional groupings which add a significant degree of variability to the data, they can in no way be considered representative of students or preservice teachers in general. As a case study, however, their responses provide insights and perceptions valid for themselves which in turn may serve to illuminate our understanding of others in algebra learning situations. Generalisability of results, then, lies in the eye of the beholder. The detail, variability and accuracy of the data serve to inform others who may then seek to discover the extent to which the findings apply to their own situation. A good grounded theory remains just that - a theory which describes and informs by the systematic and insightful way in which it was derived from a particular data set, and then invites others to provide their own experiences which may support or provide counter-examples. The latter, of course, serve a vital role in developing any theoretical position.

Grounded theory demands a specific recognition of the stance of the researcher, known as “bracketing”. Within qualitative research design, the researcher is part of the phenomenon being studied and must be aware of the values and perceptions which this inside position brings to the enterprise.

> If such bracketing is not done, the scientific enterprise collapses, and what the sociologist then believes to perceive is nothing but a mirror image of his own hopes and fears, wishes, resentments or other psychic needs; what he will then not perceive is anything that can reasonably be called social reality. (Berger and Kellner, 1981, in Hutchinson, 1988, p. 130)

Bracketing in this study is achieved through journal-keeping and research memos, kept throughout the course of the project and, specifically, through the researcher himself completing each of the Research Questions described above (providing a detailed profile), along with several open-ended mathematical tasks using available tools.

The research instrument itself provides perhaps the greatest limitation and at the same time the greatest strength of the research design. Removed as it is from the most common algebra learning mode - the mathematics classroom - it offers instead a focus upon the individual interacting with both mathematics and technology. While in no way denying the importance of social interaction in the learning processes, such a restriction potentially offers a clearer view of the issues in question. While the instructional modules created for this task may be imperfect in their realisation of the ideals of algebra learning emerging from research, they serve adequately as a starting point, or springboard, by which users may be directed and encouraged in their use of software. It is the software use which is the focus, not the nature of the algebraic environment. Further, the inclusion of open-ended problems is likely to be far more revealing of strategies of software use than the limited instructional sequences provided.

Specific strengths of the research design may be recognised in relation to the following factors:

- *Immediacy*
- *Accuracy*
- *Minimal obtrusiveness*
- *Context*
- *Reliability and Portability*

Within the context of the limitations outlined above, the computer-based learning and research environment developed for this study appears to offer certain clear advantages over alternative designs. Since the user is actually involved in an algebraic learning situation when prompted for responses and descriptions, the *immediacy* of the design potentially offers increased accuracy which may be less certain within, for example, *stimulated recall models*. These assume that certain prompts (especially audio and video recordings) will inspire accurate and detailed recollection of the thinking which occurred previously. The model developed for this study accurately captures the *flow of interaction* which is the principal object of study.

At the same time, the research instrument is designed not only to capture the responses to verbal prompts, but to monitor unobtrusively the physical interactions of individual with computer - which options are chosen, which buttons are pushed, the time taken at each card. All are accurately recorded to provide a detailed session record. In this way, the design is as *unobtrusive* as is possible within a legitimate learning situation, and certainly far more so than video and audio recording devices which intrude significantly upon the confidentiality of the learning experience.

As stated previously, the software tools under consideration in this study are always used within a particular learning context. In addition to providing instructional models developed from the results of research, the design offers an open-ended environment to which the learner may bring problems and queries of their own. This proved to be a most valuable option for the secondary students in particular, who frequently used the available utilities in a problem-based rather than instructional situation. Senior students especially preferred this mode in preparation for assessment tasks and examinations.

Possibly the most valuable feature of the research instrument lies in its *portability*. To a large extent, it operates independently of the researcher and offers the attractive option of being used at any time convenient to the participant. More importantly, it offers a level of objectivity in terms of the data collected which is more often associated with surveys and questionnaires, while retaining the flexibility and open-endedness of an informant interview.

The research design, then, offers a model of data collection and analysis which is sufficiently dense, systematic, valid and reliable to serve as a basis for the development of the grounded theory proposed.

Five
Laying out the Pieces: An Analytical Overview

In its early stages, the development of a grounded theory is analogous to the piecing together of a complex jigsaw puzzle. If the task is to be approached systematically, then it will begin with laying out the pieces, allowing them to be identified and initially sorted. In grounded theory analysis, this stage is called *open coding*, and involves the classification of the research records using codes or categories, largely arising from the data. These categories are akin to the pieces of the puzzle.

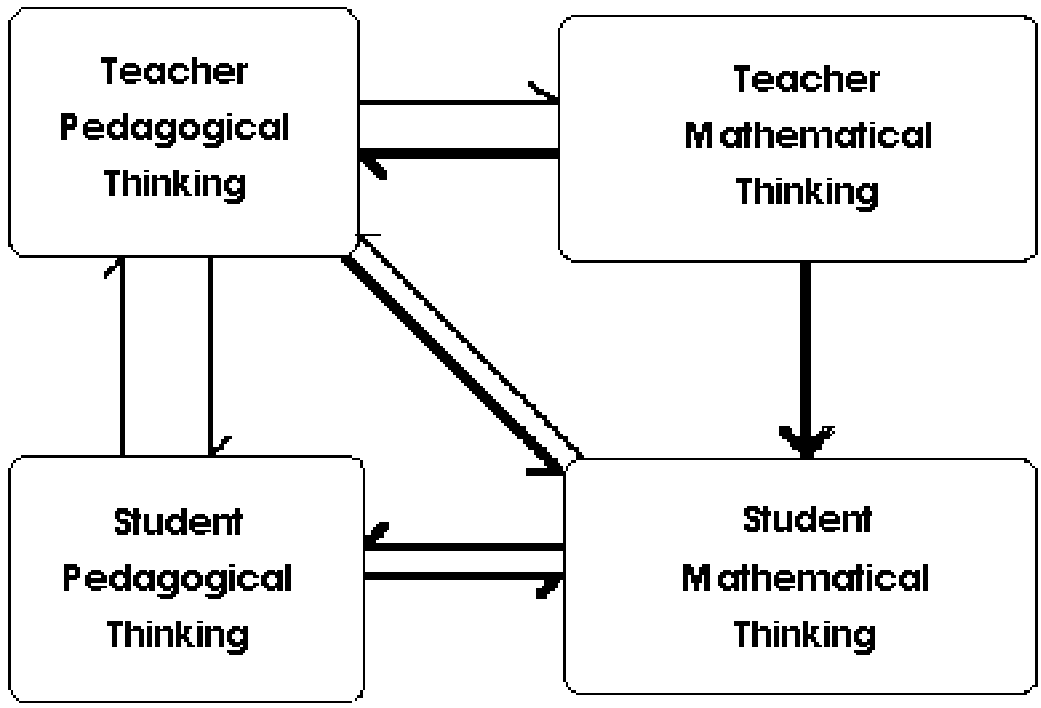
After laying them out, each piece must be studied in terms of its features, such as colour and shape. This relates to the second stage of analysis, *axial coding*, by which the individual categories are examined in terms of their properties and dimensions. Later, they will be sorted and placed in relationship to the other pieces, and the building of the grounded theory commences in earnest. Imagine now that having layed out, examined, sorted and eventually placed the individual pieces to form a coherent whole, that this whole becomes just a piece in a larger puzzle. This analogy gives some indication of the true nature of a grounded theory analysis, since it will rarely occur on a single level of complexity. Rather, it will spiral outwards as the components are examined, sorted and their nature teased out, and then they become parts of a larger puzzle which will eventually provide a rich, dense and descriptive theory of the phenomenon under consideration.

The purpose of this chapter (and the three following chapters) is to provide an analytical overview of the major categories which arose from the early "fracturing" of the data. It encompasses the two initial phases of analysis, open and axial codings. We are at the first stage of the process of building the jigsaw puzzle which will become a grounded theory of mathematical software use and by which this teacher and others might better learn to use these new tools. Prior to detailing the coding categories, however, it is relevant at this point to outline certain fundamental assumptions and perspectives which the researcher initially brought to the study, and which clearly influence the analysis which follows. Most particularly, these relate to beliefs and perceptions regarding mathematical and pedagogical thinking (the subjects of Chapters Six and Seven) and the role of mathematical software tools in the processes of mathematics teaching and learning (Chapter Eight).

An Interactive Model

Both teachers and their students engage in pedagogical and mathematical thinking, producing four identifiable domains when applied to learning situations. In the context of the present study of the use of advanced mathematical software by teachers and students, it is the *links* between these four domains of thinking which are seen to be of primary concern, providing an element of interactivity which is perceived as central to describing the processes of teaching and learning.

Figure 5.1: An Interactive Model

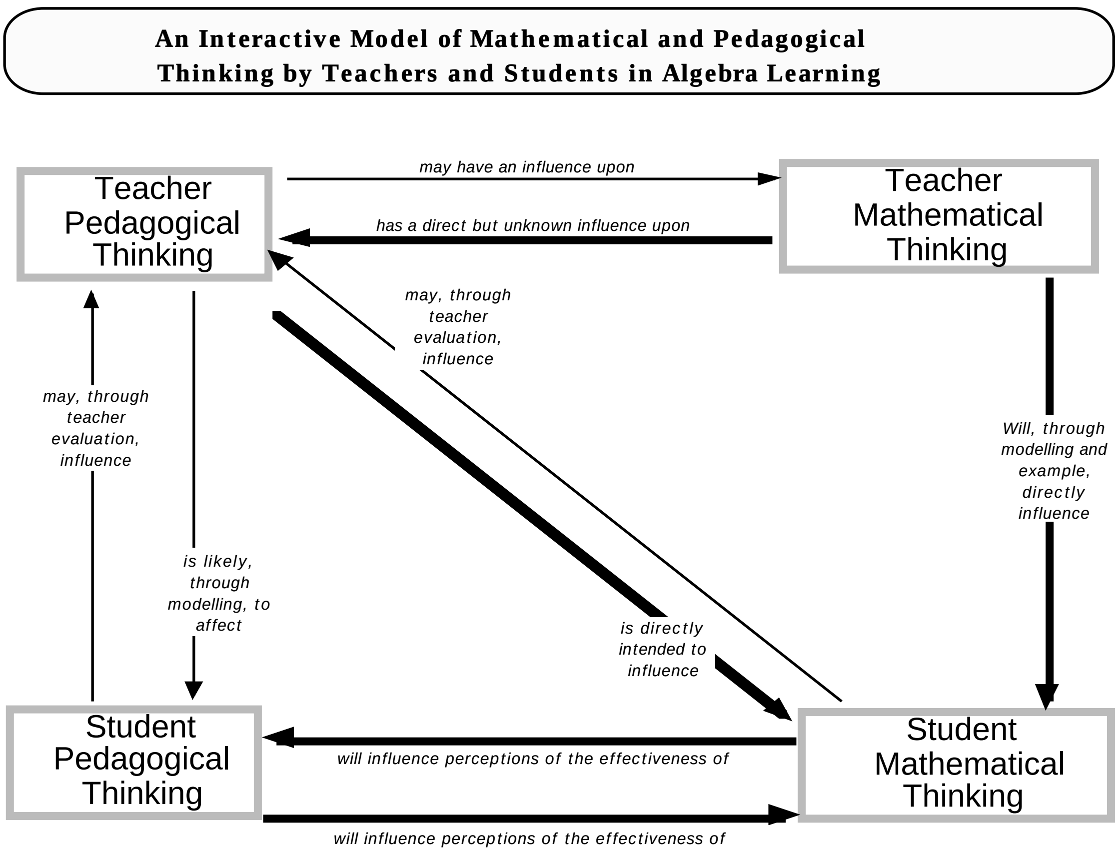


Those links shown in heavy print indicate traditional sources of focus in mathematics learning. Those in lighter print have been less emphasised and often less recognised. Each is considered below.

In the mathematical teaching/learning process, the central link is that between **teacher pedagogical thinking** and **student mathematical thinking**. This link is critical, since the primary purpose of teachers' pedagogical thinking is to directly influence the mathematical thinking of their students. Also of primary importance, **teacher mathematical thinking** is likely, through modelling and example, to indirectly influence **student mathematical thinking**. In the same way, **teacher pedagogical thinking** may, through modelling and example, influence **student pedagogical thinking** (particularly those features of the instructional environment which students associate with more and less effective learning).

The links from students to teacher are perhaps less clear, dependent as they are upon the awareness by the teacher of the students' thinking, and the willingness to respond to it. Thus, through reflection and evaluation by teachers, **student mathematical thinking** may be expected to influence the **pedagogical thinking** of their teachers. To a less clear extent, student judgements of effective method and instruction may also influence the pedagogical thinking of their teachers.

Figure 5.2: An Interactive Model of Mathematical and Pedagogical Thinking by Teachers and Students



The links between teacher mathematical thinking and pedagogical thinking have been the focus for considerable research activity (such as that by Even, 1990, 1993, Leinhardt, Zaslavsky and Stein, 1990, and Stein, Baxter and Leinhardt, 1990, in the area of “functions” alone) since Shulman’s call in 1986 to recognise the “missing paradigm” of teacher knowledge, “subject content knowledge”. At present, however, the link is still problematic, and deserving of continued study. Perhaps even more interesting may be links between the pedagogical thinking of teachers and their mathematical thinking. Teachers in general appear not to “think like” mathematicians; they think like mathematics teachers. In what ways does this affect their understanding and practice of mathematics? In the present study, it is proposed to make such links explicit through studies of the responses of teacher education students who have completed the mathematics content part of their course, but not yet studied aspects of pedagogy. Such responses (involving preferred images and levels of understanding of algebraic concepts, perceptions of effective instruction and the roles of teachers in algebra learning) will be compared and contrasted with those of secondary students.

Finally, the distinction between the pedagogical and mathematical thinking of students in schools is less clear than that for their teachers, since students are unlikely to have perceptions of mathematics beyond their classroom experiences. The two domains are certainly linked closely; although the link is unclear, each is likely to influence perceptions of the effectiveness of the other.

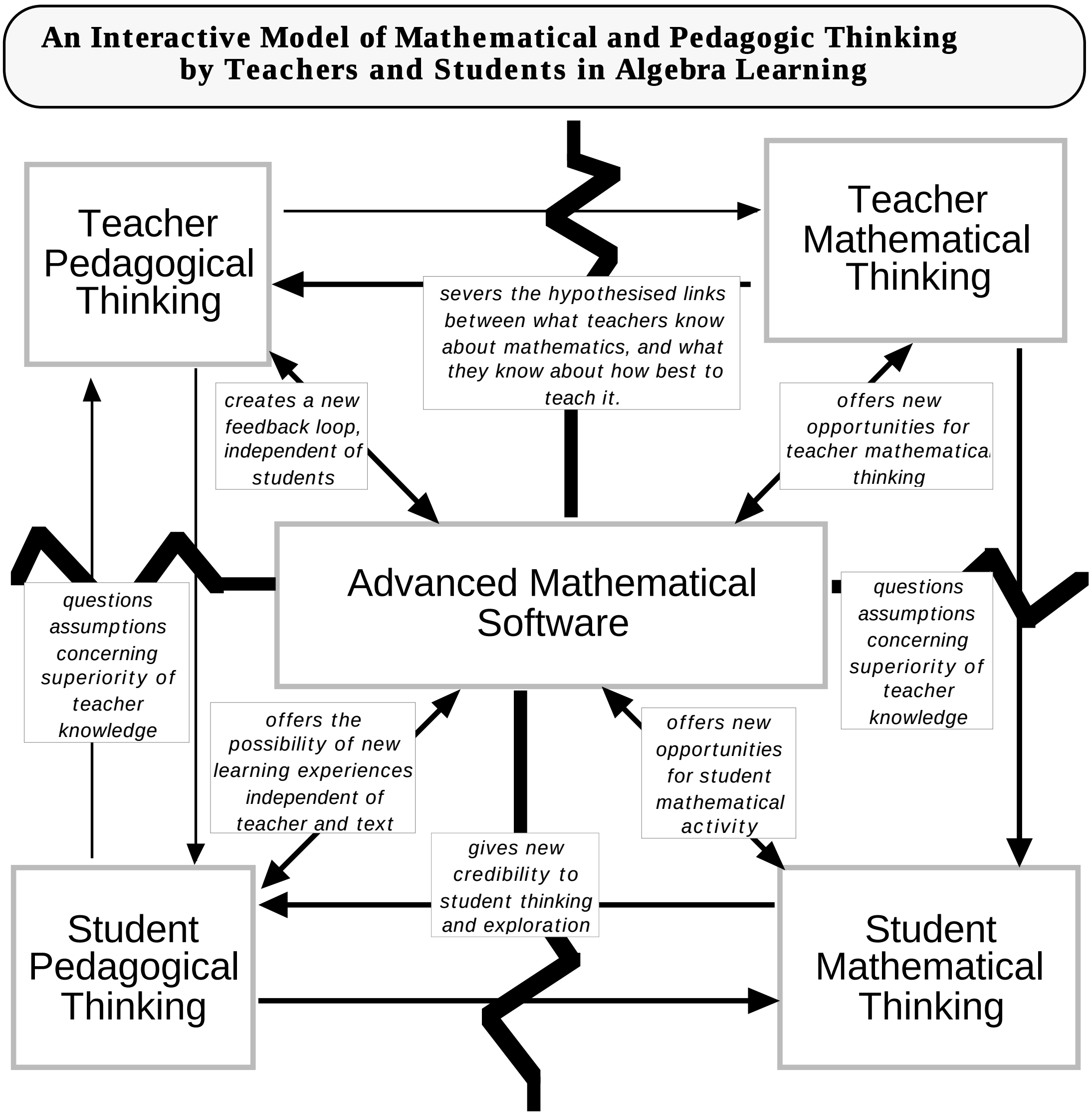
In seeking to describe and make explicit the use of mathematical software by individuals learning algebra, and its consequent effects upon their thinking, the research inquiry then must focus upon these four domains and the interactive links between them.

The Influence of Mathematical Software

In many respects, the classroom presence of computer algebra tools and other open-ended mathematical software, implies a *mediating role* between teacher and students, and between the four domains of thinking described above. It is possible to identify several possible and likely influences of such a role, and these determine the research focus which drives the present study (Figure 5.3).

The presence of mathematical software is likely to create a new feedback loop for teacher reflection and evaluation, with the software as its source. This is additional to the traditional feedback loop from students. It will also provide a new source of feedback for students, in addition to the traditional feedback offered by the teacher. It should also provide a new reference point - additional to the teacher - regarding both mathematical and pedagogical thinking. Since in many ways, teacher and students become co-learners through the use of such tools, this use is likely to question assumptions concerning the superiority of teacher knowledge of both domains. The use of computer technology in general, and mathematical software in particular, critically confronts current content and methods for the teaching and learning of mathematics.

Figure 5.3: Some Possible Effects of Advanced Mathematical Software



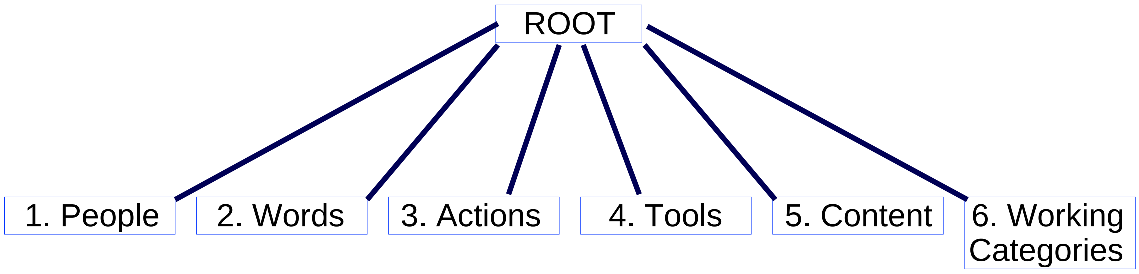
The use of mathematical software provides new opportunities for teachers to engage in mathematical thinking as distinct from pedagogical content thinking about their discipline. It is likely, too, to strengthen links between mathematical and pedagogical thinking by students: The use of mathematical software provides new opportunities for students to engage in mathematical thinking which is independent of teacher and text, and so gives new value to mathematical creativity and investigation. It may serve to provide new validity for student mathematical conjecture, and to give new credibility to student thinking about mathematics, no longer wholly subject to the teacher.

Having made public these perceptions and beliefs, it is now possible to approach the analysis of the data with some awareness of the researcher's position in constructing the initial index system. This initial overview will be structured around the three central concerns of the project - mathematics, pedagogy and computer use, each analysed in terms of the components which are outlined below.

Categories of the First Level

The qualitative analysis software program, *NUD•IST*, encourages the sorting of conceptual categories into logical tree structures as they arise from the data. Each category occupies what is termed a **node** in the tree which forms the **index system.** Beginning with a "root" node, all others are placed in relationships which will initially reflect the cognitive organisation of the researcher (Figure 5.4). Later, these nodes will be moved and reorganised to better reflect the structure of the data and the unfolding nature of the grounded theory. At this stage, however, it suffices to examine these categories within the descriptive and largely superficial relationships ascribed to them by the researcher, and so to better understand the perspective and biases which he brings to the analysis. As mentioned previously, this process of "bracketing" is essential if the theory is to develop from the data and to be more than a mirror of the researcher's own views.

Figure 5.4: Categories of the first level



Five of the six first level categories displayed in Figure 5.4 reflect the initial analysis of the project into what were perceived as its critical analytical components (category 6, *Working Categories*, provides a functional repository for new categories as they arose from the data, allowing their placement in relation to the others to be delayed until later in the process. These would generally be categories which did not fit readily into the existing structure).

Figure 5.5: Category 1: People

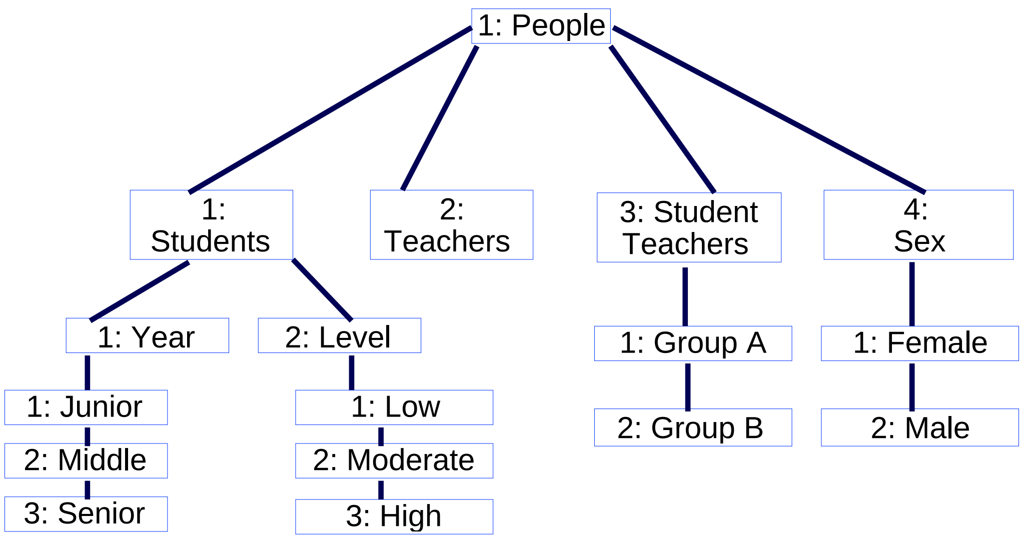
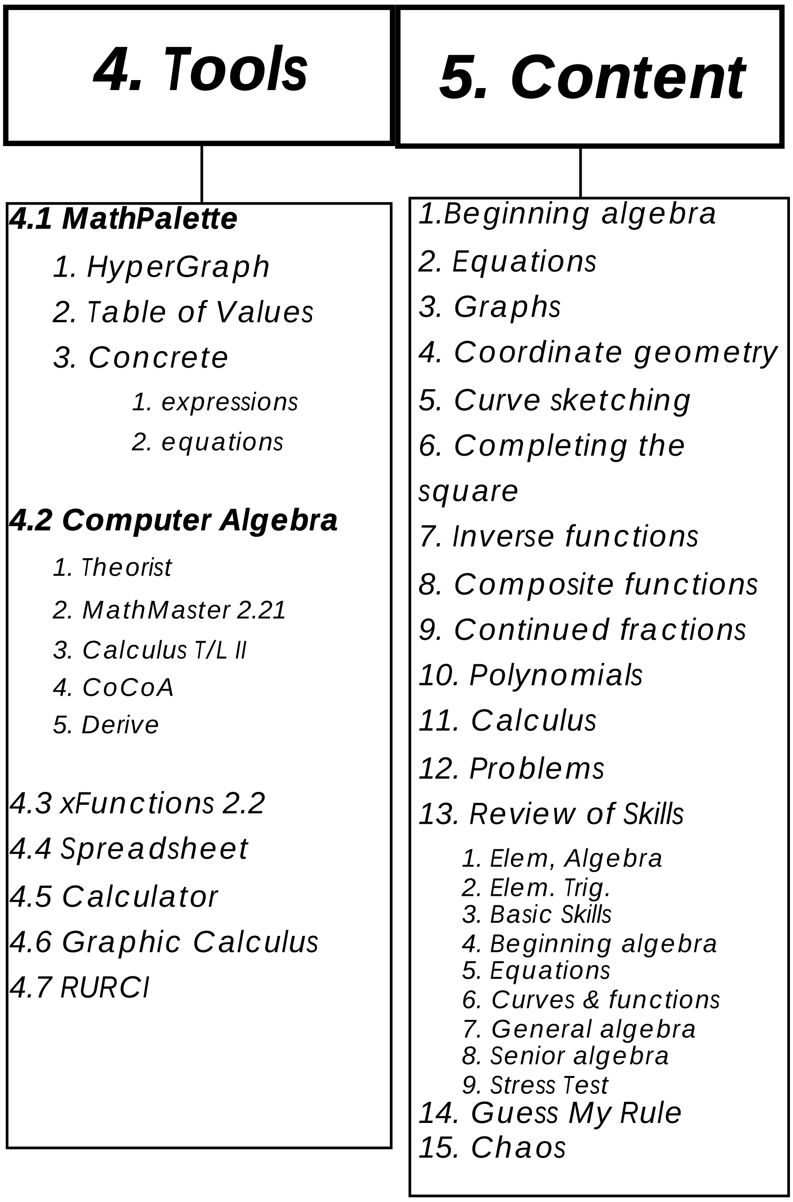


Figure 5.6: Tools and Content Categories

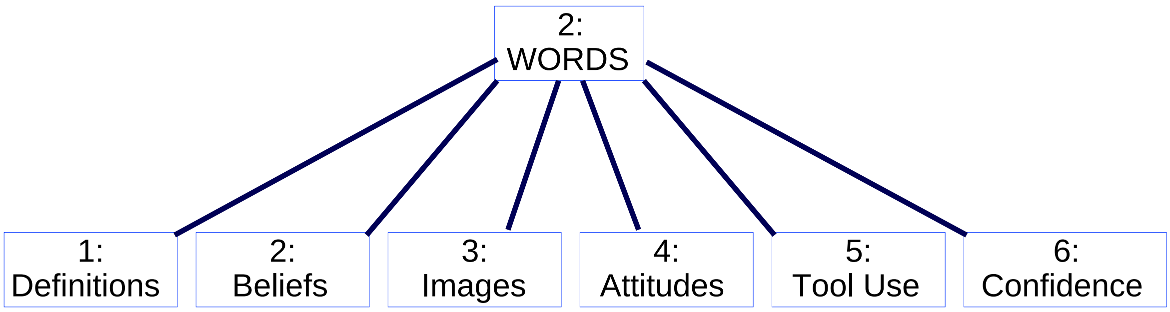


The categories *People, Tools* and *Content* are purely descriptive in nature, reflecting aspects of the physical components which make up the study (Figures 5.5 and 5.6). These have been discussed in previous chapters. (Note that the tool "Graphic Calculus" refers to an IBM-based package by David Tall, *A Graphic Approach to the Calculus*, which was used early in the gathering of data, and *RURCI* stands for *Are you Ready for Calculus?*, a series of review quizzes compiled for IBM computers by David Lovelock, which were later adapted into the *HyperCard* format of this study.)

The two remaining categories, *Words* and *Actions*, represent major units of analysis for the data, since they are perceived as the means by which *thinking* is made explicit on the parts of the participants. The ongoing development of these groupings occupied the primary focus in the early coding, both open and axial, and the description of each that follows provides the greater part of the overview for this chapter. These two nodes capture in detail the nature of the interactions which later provide the basis for the relationships which comprise the grounded theory. In particular, the category, **Words**, provides the basis for the analysis of the three core categories - thinking about algebra, about pedagogy and about computers, which occupy the next three chapters, and so lay the groundwork for the subsequent theory generating process. Consequently, it is examined here in some detail.

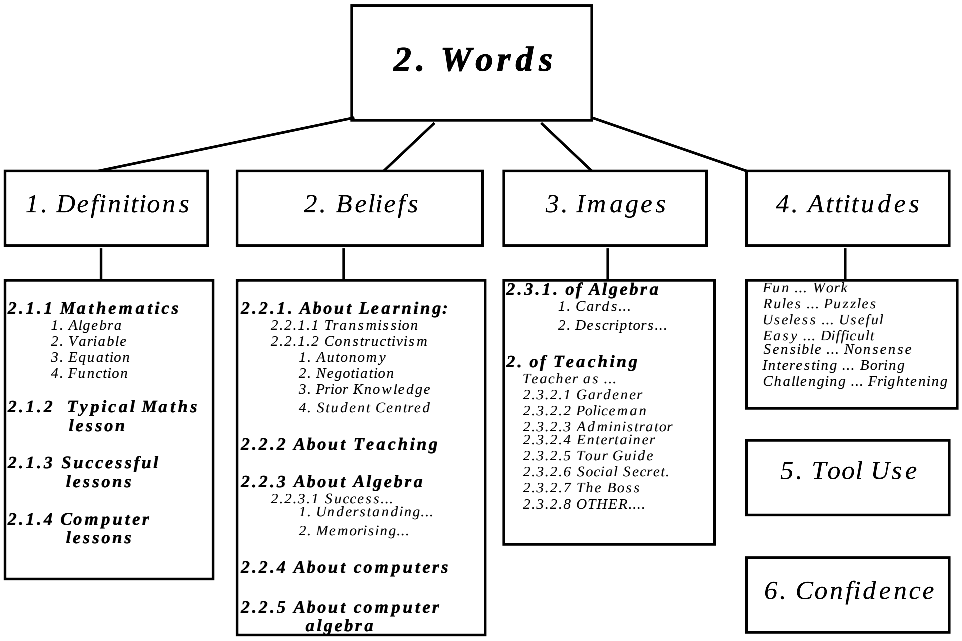
A Study of Words

Figure 5.7: Category 2: Words



The initial conception of the category involved three children, **Definitions, Beliefs** and **Images. Definitions** are seen as objective attempts to describe concepts, as opposed to **beliefs**, which may be expected to contain a subjective element, as the object is described *in relation to* the person giving the description. Within a constructivist framework there can be no definitive boundary between the two conceptions, as all knowledge is constructed from personal meanings - every "definition" represents a statement of personal belief. However, for the purposes of this study, a distinction between the two is useful, and to this end an objective/subjective distinction should suffice. The focus question, *"How would you describe algebra and the way you best learn it?"* might be considered to give rise both to a definition (*"Algebra is..."*) and a statement of belief (*"I best learn algebra by..."*). The first is seen to exist independently of the speaker, while the second is defined in terms of the relationship with the speaker, and usually involves an explicit value judgement (for example, "How do you *best* learn algebra?"). Although beliefs may be "messy constructs" to examine subjectively, "few would argue that the beliefs teachers hold influence their perceptions and judgements, which, in turn, affect their behaviour" (Pajares, 1992, p. 307). Nor may this influence be restricted to teachers. Lying at the intersection of the cognitive and affective domains, "people's conceptions of mathematics shape the ways that they engage in mathematical activities" (Schoenfeld, 1989, p. 338). The beliefs individuals hold regarding mathematics, algebra, computers, learning and teaching must be considered critical issues in the present context.

Figure 5.8: Sub-categories of WORDS



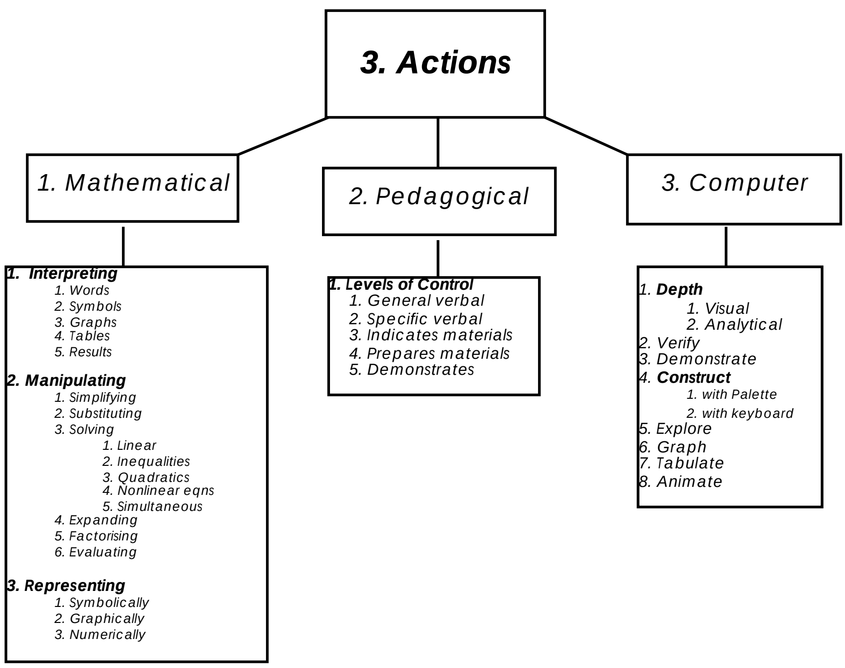
If beliefs may be distinguished from definitions by their subjectivity and their inclusion of a value dimension, then what of images? Vinner defines a "concept definition" as "a verbal definition that accurately explains a concept in a non-circular way" (Vinner, 1983, p. 293). While this very "definition" is itself circular (defining a definition as a type of definition), it offers three critical elements - a definition may be considered to be *verbal*, it *explains* something and it should be *non-circular*. It is probably the first of these which most clearly distinguishes concept *definition* from concept *image*, which Vinner describes in terms of one's *mental picture* and the set of associated properties called up by the concept. In SOLO terms, the concept image is ikonic: visual, intuitive and global, while the definition is more likely to be associated with concrete-symbolic thinking - rational, sequential and verbal. Van Hiele's distinction between *visual* and *descriptive* levels of thinking applies equally.

These three analytical units, definition, belief and image, provide the primary basis for analysis of the research data (Figure 5.8). Taken together, they allow rich and detailed depiction of elements of thinking related to both mathematics and pedagogy within a tool-rich algebraic learning context. As the analysis of the data unfolded, however, three additional categories arose, each of which reflected important considerations which did not fit easily within the early codes. Statements relating to attitudes, tool use and confidence were considered particularly significant in the study, and were positioned accordingly.

A Study of Actions

As illustrated in Figure 5.9, actions were categorised in terms of the three central concerns of the study - mathematical, pedagogical and computer-related actions. The sub-categories for each will be examined in detail in Chapter 8.

Figure 5.9: Categories of Action

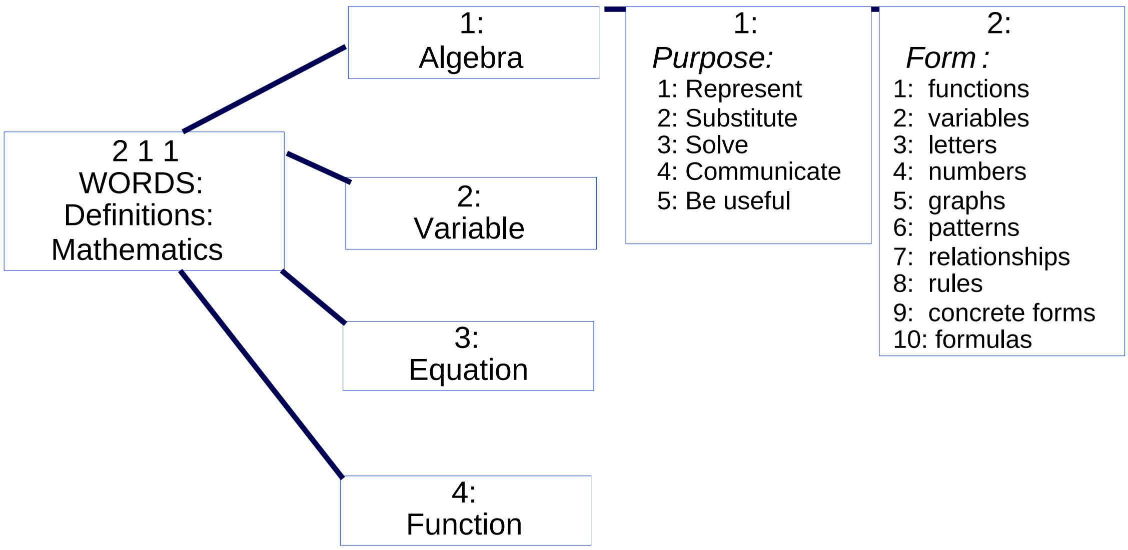


Six
Thinking about Algebra

Definitions of Algebra

The mathematical definitions elicited within the study centred upon four elements - definitions of algebra itself, and of three central constituents of algebra - variable, equation and function. The data gave rise to an analytical feature common to all four definitions - the distinction between **form** and **purpose** (akin to Piaget's **structure** and **function**). Responses identified with **form** were those associated with attempts to describe the nature of algebra in terms of its constituent parts. These included "a group of letters that may stand in place of numbers" (S3: Jane), "algebra is many numbers but expressed as letters" (S1: Andrea) and "algebra is an equation or rule" (S4: Stephen). Each reflects a perception of what algebra *is*, as opposed to what algebra *does*. This latter is associated here with a conception of **purpose**, and is reflected in definitions such as "algebra is finding or solving or using unknowns to get a correct solution, it is like putting one number in then getting another number out to help solve the problem" (S1: Andrea). Within both these distinctions may be found a range of perspectives which make up the sub-categories depicted in Figure 6.1.

Figure 6.1: Definitions of mathematics



The form or structure of algebra involved the identification of ten different descriptors. **Functions** were mentioned explicitly only by the researcher ("The principal concepts of algebra are functions and variables - functions defining a relationship between variables which can be expressed in various representations" (T1: SMA)) and two of the student teachers. Both of these were in a subsidiary context to the concept of variable (A3: "Through the use of their interpretation and analysis skills of graphical presentations, students are in the position to gain insights into the dynamism of functions and variables," and A4: "students can see how the letters can represent a function".) The emphasis placed upon the concept of **variable** in their preceding course related to algebra teaching and learning was clear in the responses of the Group A preservice teachers. All saw it as important that students "develop an understanding of variable and the relationships between variables in Algebra" (A1) and "for students to be successful in this introductory stage of algebra learning, and in fact, all other stages, they must develop an understanding of what a variable is" (A3). Each of the preservice teachers from Group A who made an attempt to define algebra included specific and explicit mention of the central role of understanding of variable. None of the secondary students made such explicit reference to either functions or variables in their definitions of algebra.

For the secondary students, the nature of algebra was defined by **letters** and **numbers**, with less frequent references to **rules, formulas** and **graphs**. An early interview with Jane (S3) as a Year 10 student studying Advanced level mathematics revealed a shallow understanding of this nature:

Interviewer: .	..This person, towards the end of the lesson, nudges you again … and says, "What's algebra?" We've had this one before, but how would you explain it to someone who didn't know?
Jane:	Letters?
Interviewer:	Then he says, "Letters?" Yeah, that's what we do in English.
Jane:	Um … numbers and letters?
Interviewer:	Okay…
Jane:	Um … a group of letters that mean something, equal something?
Interviewer:	Okay, a group of letters that mean something, equal something … equal what? Equal numbers?
Jane:	Yeah.
Interviewer:	Okay, so a letter like "a" can stand for a number?
Jane:	Yeah.
Interviewer:	Okay, can it stand for more than one number?
Jane:	Yes.
Interviewer:	So it can stand for what? Two numbers?
Jane:	It can stand for … I don't know … anything.
Interviewer:	Anything?
Jane:	Any numbers.

This transcript is revealing of the tacit nature of knowledge about algebra. It is clearly disjointed and difficult to articulate for the student and yet, upon probing, is essentially sound in its foundations. The important concept of variable as representing a range of values rather than simply a "placeholder" is present here, suggested at the end of the transcript. It was as a consequence of this recognition of the tacit nature of students' knowledge about algebra that alternative ways of gathering research data were explored, leading to the image-based methods developed later in the study.

Table 6.1

Forms of Algebra

Forms	function	variable	letters	numbers	graphs	patterns	relations	rules	concrete	formulas
A1		√				√	√	√	√	√
A3	√	√		√	√	√		√	√	√
A4	√	√		√		√		√	√	√
A5		√						√	√	√
A6		√						√	√	√
S1			√	√				√		√
S2					√			√		√
S3			√	√						
S4			√	√	√	√		√		√
S5			√	√						
S6									√	
T1	√	√					√			

Table 6.1 presents a summary of responses to the nature of algebra which reflect a consideration of its constituent components. Several subgroups may be discerned from these responses. Patrick (S6 - Year 7) might best be termed *pre-algebraic,* his only experience of algebra at this stage being informally, through concrete models. Jane (S3 - Year 10) and Tony (S5 - Year 8) are both limited to a *unistructural* understanding of algebra, as "letters standing for numbers". It would appear that entry into senior mathematical study brings with it a broader conception which commonly includes recognition of the roles of symbolism (equations and formulas) and relationships in defining the nature of algebra. Such an understanding, however, appears at best *multistructural*, even among the preservice teachers, as their understanding is made up of a collection of largely unrelated parts. Understanding of the relationships between the various components of algebra - the letters, the numbers, the graphs, the rules and formulas, even the place of functions and variables - appeared to be largely absent among both students and preservice teachers, at least as revealed by the verbal definitions offered.

Five dimensions of algebraic *purpose* were also identified from the data. Most common of these was the perceived function of **solving**, found among all the older students (Andrea (S1), Ben (S2), Jane (S3) and Stephen (S4)) and two student teachers (A5 and A6). These definitions included "algebra is a form of mathematics which helps us in ways to solve mathematical problems" (Ben), "we use algebra to find out numbers that we do not know" (Tony), and "I believe algebra to be a mathematical expression for finding unknowns" (A5).

Another common perception of the purpose of algebra was its **representative** function - most commonly as letters standing for numbers (already mentioned above), but with other references to **graphs, patterns** and **concrete forms** as serving representational roles. Among the secondary students, there was evidence of a shift over time, from a **static** view of algebra (aligned with its representative function) to a more **active** view (associated especially with **solving** problems). Jane's early definition of algebra, captured in the interview transcript above, is persistent over time ("letters and numbers", "letters that stand for numbers" and "algebra here means anything with letters in it") until finally, it includes a new component: "Algebra is a group of numbers and letters to solve equations". This added dimension of purpose in addition to form appears to suggest some change in perception over the period of the study.

Jane's passive view of algebra may be contrasted with a more active view, such as that consistently displayed by Stephen (S4). From his earliest definition ("Algebra is a topic of maths which has letters and numbers. Solve these equations [sic] and simplifying them is the main target. Algebra is no set form and can be described on graphs or number lines or other ways".) Stephen clearly displayed a more comprehensive view of algebra than that of Jane, a view that was similarly persistent over the period of the study: "Algebra is a way of understanding a[n] equation with an unknown value used to describe the pattern it makes. It is useful to know how to deal with algebra so you can know how to understand statements without knowing its value". Compare this early definition with later ones: "Algebra is an equation or rule which is a guideline to answering certain types of equations which involve pronumerals” and “Algebra is a way of describing a graph or equation with pronumerals and numerals”.

Stephen’s quite comprehensive definitions may be compared with those of his peer, Ben (S2). While Stephen was attempting the higher level Three Unit mathematics course for the Higher School Certificate, Ben was attempting the Two Unit course. For Ben, “Algebra is a form of mathematics which helps us in ways to solve mathematical problems. It’s all done with formulas and visual aids - graphs, plotting in the number plane, curves and different equations that you can graph”. Much later, this had been refined to “Algebra is solving algebraic problems (graphically, algebraically) where there is always an unknown”. For Ben, algebra is about solving, whereas Stephen’s view included a clear descriptive function as well.

Andrea (S1), too, shared this perception. Developing from her early representative depiction, “algebra is many numbers but expressed as letters”, her view of algebra grew over time to become far more active: “Algebra is finding or solving or using unknowns to get a correct solution”. Like Stephen, Andrea was attempting the higher level Three Unit course in Year 11 at the time of the study. Their definitions appeared to be revealing of a broader conception of algebra than that of Ben.

As a Year 8 student, Tony (S5) might be expected to display quite limited understanding of the nature and role of algebra. It is interesting, then, to note that even his earliest attempt - “Algebra is where you use letters to substitute for numbers” - a clear element of action was evident, and such a view was labelled **substitution** to distinguish it from the passive **representation** form displayed by others, such as Jane. A much later definition showed the twin elements of representation and solution which had also categorised the developments of the older students:

> Algebra is the use of letters in mathematics as pronumerals for numbers that we do not know. We use algebra to find out numbers that we don't know. An equation is using algebra to find out the value of a pronumeral.

Although Tony was graded at this time as a middle ability student at his school (placed in the third of five graded mathematics classes), his responses seem indicative of a higher ability range than this. (Towards the end of the study, Tony had actually been promoted two class levels, suggesting that student understanding of mathematical concepts may be a useful indicator of mathematical capabilities. At the same time, the ability to express oneself verbally in a clear and articulate way may also be a relevant indicator in this context, possibly independent of mathematical ability.)

This function associated with a process of "solving" was commonly linked to another, **being useful**, in which explicit recognition was made of algebra serving some helpful role. Although the *purposes* of algebra were categorised in a variety of ways already discussed, few mentioned any *application* of algebra beyond its own ends. In other words, for most participants, the purpose of algebra is to solve algebraic problems. Of the students, Andrea noted that "we do algebra to make life easier - without algebra we wouldn't have half the stuff we do now, or maths", while Tony suggested that "it helps us in everyday life when adding stuff up at the supermarket (a basic example) and all kinds of other ways" (although he may well have been referring here to mathematics in general). Even the preservice teachers, at the end of their formal studies of mathematics and pedagogy, were able to offer only that "algebra is a highly useful area of mathematics that is a crucial foundation to much of the mathematics that High School students learn" (A6) and "algebra is important to pupils learning of mathematics. It is the basis of the maths pupils will do in later years". (A2).

Only one participant (A3) offered an alternative view of algebra, as a means of **communicating**:

> Students need to view algebra as a language, a way of saying or communicating a rule in an abbreviated form. The use of symbolism, the abbreviated form, incorporates the idea of generalisation.

This view hints at the potential of algebra as a mediating tool which supports thinking and enables higher cognitive functioning in the same way as language. It was evident nowhere else in the data, however, and is clearly not a general perception of algebra among the participants.

The definitions of *equation, variable* and *function* were similarly analysed into statements related to **form** and **purpose**. The sub-categories for each are summarised below, along with those participants from whom these categories were derived.

Thinking about Equations

Equation is…	
(1) Purpose	*(2) Form*
• To represent:	• manipulations (A1, A4)
- equality (S3, S4)	• representations:
- relationship (S5)	- graphical (A1, S4)
• To solve (S1)	- symbolic (A4, S3, S4)
	• variables (A1, S1, S5)

The familiar identification of equations with the manipulation of symbols to produce an answer was explicitly found only among the student teachers. For the students, an equation was more likely to be defined in terms of graphs and symbols arranged around an equality. While the notion of equality and the purpose of deriving an "answer" were generally recognised properties, the means by which this answer might be derived was notably absent from consideration. Once again, some responses among the students were clearly **unistructural** ("a set of numbers and letters that equal another set" (S3: Jane) and "when you make a rule or relationship between numbers and pronumerals" (S5: Tony), while others indicated thinking which was **multistructural** ("An equation is a set of numbers which might contain variables - when you put one number in, another number will come out. e.g. x + 1 = y (two variables)" (S1: Andrea) and "an equation is a statement which describes a line or curve which can be drawn on a graph. It includes a group of numbers with letters and their values. You try and get the letters (usually x and y values) to equal a number" (S4: Stephen).

Comparison of the two responses classified as unistructural reveals that, while both focus upon a single property, they are nonetheless quite distinct in nature. A similar finding occurs with the second pair of examples. In both cases, the second response illustrates a higher cognitive level than the first. Consider, for example, the two responses classified as unistructural. The object of focus for the first is a static notion of equality. It views the equation as a totality and appears to illustrate what van Hiele terms a “visual” response. The second response, however, takes as its focus the establishment of a relationship, an active perception quite different to the first. Similarly, the second multistructural response is far more complex than the first. These examples appear to support the recent developments in the SOLO taxonomy which suggest that, within a single mode of thinking (in this case, concrete-symbolic), there may be found several cycles of increasing complexity, rather than the single unistructural-multistructural-relational cycle originally proposed (Pegg, 1992).

Perhaps even more illuminating is the distinction drawn by van Hiele between what he terms the **symbol** and **signal characters** of cognitive objects such as geometric figures or, as in this case, equations and algebraic symbols (van Hiele, 1986, pp. 60-61, p. 168). Initially, he proposes, an object is recognised by its **symbol character** (van Hiele, 1986):

> Many symbols begin their existence with an image in which the observed properties and relations are temporarily projected. However, after the explication of those properties and relations by an analysis or discussion, the symbol loses the character of image, acquires a verbal content, and thus becomes more useful for operations of thinking. (p. 61)

Such a view is likely to be visual in nature - global, wholistic, intuitive. Gradually, however, van Hiele submits that the symbol acquires new properties - "the symbols act as signals" (van Hiele, 1986, p. 62). As such, they influence the thinking of the individual in a direct and deliberate way. As signals, they will trigger a cognitive reaction, which may be a recognition of a complex of properties and relationships or may even be a *signal to act* within a specific context.

The example of equation is an interesting case in point. When an individual views an equation, what is actually seen? At the lowest level (in a *prealgebraic* sense) an equation is simply a jumble of letters and numbers (similar to Jane's response above). Having acquired a symbol character through early study of algebra, it is recognised as defining a relationship of equality (Tony's response). (This does not imply that Jane was operating at a prealgebraic level, only that her response might be seen as illustrative of such a level).

If the second pair of responses are considered, both illustrate recognition of signal characteristics of equations. Much of the training in early algebra is intended to produce a particular signal response in students - an automatic triggering of the signal to act in a predetermined sequence which will eventually result in a solution. Both multistructural responses above display this signal nature - they recognise that an equation is an object to be used to produce a result, an answer. Stephen's response, however, appears to go one step further. His view of equation triggers not only the signal to act, but also the recognition of other representations of equation which are indicative of a richer network of relationships.

What an individual sees when confronted by an equation, a function, a graph or any of the symbolic objects associated with algebra must be recognised as a central issue in the present study. Further insights into such thinking are likely to arise from consideration of responses to visual images of algebra, and it seems likely that the use of computer tools must be considered in the context of such visual thinking.

Thinking about Variables

Variable is...

(1) Purpose	*(2) Form*
• To represent:	• pronumeral (A1,A3,A4,S1,S5)
- unknown value (S1, S4)	• patterns (A1, A4, A5)
- numerical value (A2, A3, S4, S5, S6)	• rule (A2, S4)
- range of values (A1, A3, A4, A5, A6, S4, S6)	• dynamic (A2, A3, A5, S4, S5)
	• letter (A3, S4, S6)
- lines/curves (S4)	
• To solve (S4)	
• To simplify (S6)	

Understanding of the concept of variable has been a common focus for studies associated with the learning of algebra (for example, Quinlan, 1992) including learning within a computer-based context (Boers, 1992). In the latter case, it was found that students with access to computer algebra software were more likely to think of variables as representing a range of values than as a single placeholder for an unknown object. Such was found to be the case in the present study, where student definitions of variable tended to be active, process-oriented conceptions: Stephen (S4), for example, thought of variable early in the study as “a letter used in algebra which is used to describe certain lines or curves or to use as an unanswerable value. Most variables are able to be used to find answers to... They are mostly used in rules of finding answers". Later, however, a variable was "how you change the answer, by using different numbers". Andrea (S1), too, described variables in terms of multiple unknowns: "a symbol representing an unknown anything - usually numbers, or a degree (angle)".

Of considerable interest were the responses of the two junior secondary students, Patrick (S6) and Tony (S5). Tony had not encountered variables formally outside his use of the computer-based instructional modules developed for the study. His active perception of variable as "a changing amount of numbers or pronumerals in a sum, equation or whatever" may be attributed to his computer-based learning context. Patrick, even more so, had studied no algebra, and had worked through the *Beginning Algebra* modules and used the *Concrete Algebra* modes available. His comments display a firm and clear understanding of the concept of variable within the symbolic context of algebra:

> My theory about variables is that by replacing words with single letters it makes maths easier for those who have troubles with long words. The letters stand for amounts of things and numbers ... [A variable is] replacing amounts of substances or numbers with single letters. It can stand for either one or many numbers.

Later, when describing the meaning of given concrete shapes to which had been assigned letters representing their areas, Patrick noted:

> 'm' stands for any shape or form which covers five squares, m = 5.
> 's' stands for any shape which covers six squares, s = 6.
> 'a' stands for any shape which covers two squares, a = 2.
> At the moment, m = 5 but this is not a permanent fixture as it is only this at the time. And the same rules apply for other letters.

While it is hardly surprising that the student teachers displayed consistent and versatile understanding of the concept of variable (since it had been an area of particular focus in their previous studies), the depth of understanding of the concept displayed by the secondary students appears to derive to a great extent from the technology-rich algebra learning environment which they had shared, within which the attainment of an active process conception of variable had been an explicit priority.

Thinking about Functions

Function is...	
(1) Purpose	*(2) Form*
• To represent:	• graph (A1, A3, A5, S3)
- unique value (A1- 6)	• rule (A1, A3, A4, A6, S4)
- non-unique value (S4)	• input/output machine (A1, A6, S3, S4)
- action (A1, S3)	• domain/range (A1, A2, A3, A5, A6)
• To solve (A4, S1, S4, S6)	• table of values (A3)
	• equation (A5, S1, S4)
	• unknowns (A4, A5)
	• patterns (A5, S4)
	• set of numbers (A5, S3)
	• relationship (S4, S6)

In addition to the concept of variable, the other area of particular focus in the construction of the computer-based learning environment for the study was an understanding of **function**. The emphasis within the program was upon building a **versatile** conception of function, within which students would have access to a relatively rich cognitive repertoire when considering functions.

The preservice teachers displayed a thorough knowledge and understanding of this concept. All acknowledged the formal requirement for uniqueness which, although specifically mentioned within the instructional modules of the program, did not appear among the definitions of the secondary students. The latter were more likely to associate function with **solving**, arising from a perception that functions and equations were essentially the same. This confusion is evident in Andrea's definition:

> A function is a y value of an equation but not the y value when the equation equals zero. You can simplify a general form equation, however, you cannot do this to a function even though the numbers are exactly the same. e.g. f(x) = 3x^2+6x+9 is a function.

This confusion was more pronounced for Jane, who really did not know what a function was, but suspected that it was probably very similar to an equation:

Interviewer:	Well ... in your way of thinking, would that be a ... is that what you would think of as a function?
Jane:	[Long pause] No. I don't know. Um...
Interviewer:	So you're not sure?
Jane:	No.
Interviewer:	That's fine. Do you reckon they would mean the same thing?
Jane:	Yeah.
Interviewer:	So they're both really like equations?
Jane:	Yeah.

It seems likely that Jane was grasping at straws at the end of this interview, trying to escape from a difficult situation where she was being questioned about something which she really did not know. While she was familiar with the term from her mathematics classes, she really showed no understanding of its nature. She did, however, identify function, not only with equation, but with ordered pairs and with plotting sets of points onto a graph, suggesting that her mental image of function was perhaps richer than her verbal definition.

Stephen also displayed a diverse conception of function, which included number patterns, input/output machine images and the association with a rule or relationship. His understanding of function developed significantly throughout the period of the study. His early thinking was unistructural (although clearly active): "a function is [when] a value of one is determined upon a value of another". His response to the symbol $f(x)$ was "the values that x can be in the equation written after it ... and you put numbers for the x-values, and use them to work out what that equals". Stephen was quite definite about the distinction between functions and equations - there was none.

Interviewer:	Alright, last question. This is the easy one. What's the difference, if any ... is there any difference between functions and equations?
Stephen:	No.
Interviewer:	Right. So you gave me an example of a function before which was f of 1 equals... or f of x equals x cubed minus whatever. Give me an example of an equation.
Stephen:	x squared minus 6x plus three ... equals zero.
Interviewer:	Alright, alright, so what's the important thing about an equation?
Stephen:	It equals zero, and it also has to have a number ... at the end to show where it crosses the y axis, or whatever.
Interviewer:	Alright, so, say you've got ... y equals 3x plus 1. Is that an equation? Is it a function?
Stephen:	Yes, it can be.

Interviewer:	Good. So it doesn't have to have the 'f(x)' part to make it a function? Which part would you say was the 'function part', or is the whole thing the function? y equals 3x plus one.
Stephen:	Uh, the function is ... x ... the part after the y.
Interviewer:	It is what it equals?
Stephen:	Mm.
Interviewer:	What about something like 'x plus y equals two'? Is that an equation?
Stephen:	Mm.
Interviewer:	Why?
Stephen:	Because it's got both x and y values, and it equals a number. But in a function ... it's ... the x value plus a number ... with y is the function out the front.

This interview is revealing of the fragility of understanding of basic algebraic concepts, even by students considered quite mathematically capable. Even the most common of algebraic entities, the equation, appears to present a minefield of uncertainty for students. The transcript suggests, too, that the physical arrangement of algebraic forms plays a very important part in student perceptions - Stephen's insistence that a function requires a part "out the front" is significant, particularly within a computer-based context where, for example, many graph plotters require functions to be entered in a specific format (usually "y =" or even "f(x) ="). The *HyperCard* plotter and table of values utilities developed in response to these interviews deliberately allowed functions to be entered with or without a "y =" prefix, in order to study student preferences and any potential effects upon student thinking about functions.

The interview continued, attempting to further tease out Stephen's thinking about functions and equations:

Interviewer:	... So what about something like... sine of x? Now is that a function?
Stephen:	Yeah ... mm... [unsure]
Interviewer:	Wouldn't like to put a hundred dollars on it?
Stephen:	No.
Interviewer:	Alright, is it an equation?
Stephen:	Yes.
Interviewer:	It is?
Stephen:	Just sin(x)?
Interviewer:	Sin(x).
Stephen:	No.
Interviewer:	So what would it need to be an equation?
Stephen:	y equals or ... something like that.
Interviewer:	To have it equals something?
Stephen:	Mm.
Interviewer:	Alright, last one. What about ... x equals 4.
Stephen:	That's an equation.
Interviewer:	Alright, is it a function?
Stephen:	uh ... no.
Interviewer:	Okay, so ... how would ... could you make it into a function?
Stephen:	Yeah, just ... x minus four equals something ... a y or zero.

Stephen's thinking about functions does not include the formal uniqueness property. It corresponds instead to a general notion of a rule or relationship which must be presented in a specific format. Thus, "x = 4" is not a function to Stephen, not because it fails the vertical line test, but because it is not written the right way.

Stephen's thinking about functions became more clear during the course of the project. Soon after the initial interview, he was able to state that "a function is a statement or rule [in] which you can use for any numbers to find an answer which comes out from using the function. [For example] f(x) = x^2 - 4x + 3, f(x) = 3x - 11". Much later, towards the end of the data gathering phase of the study, his thinking had clarified even further:

> A function is a way of describing a certain pattern. When substituting various numbers into it, it will give different numbers which have been changed according to the value of the function. f(x) = x^2 - 2, f(x) = 3x - x^2.

Stephen's early disjoint conception had solidified to a clear and versatile understanding of both nature and purpose. Later study of the images used to think about these concepts, however, revealed that, even at this stage, inconsistencies still existed within Stephen's understanding of these central concepts.

Of the two junior secondary students, after an introduction to the ideas of function, Tony remained unable to articulate a verbal definition - "I can do the questions but I don't know what it is," while Patrick demonstrated a more coherent grasp of the concept. Asked to try to give two different explanations of what a function is, he offered that "a function is starting off with a difficulty in the sum and then working out what the characters in the sum would need to do to solve it" (probably identifying function with equation), but then noted that "a function is being in a way related to someone or something," demonstrating a fundamental grasp of the central idea.

Summary

The various definitions of algebra, then, suggest a coherent and consistent view which appears to permeate all levels studied. An early understanding of algebra is most likely to be static, fixated upon its representative nature, defined by the use of letters replacing numbers.

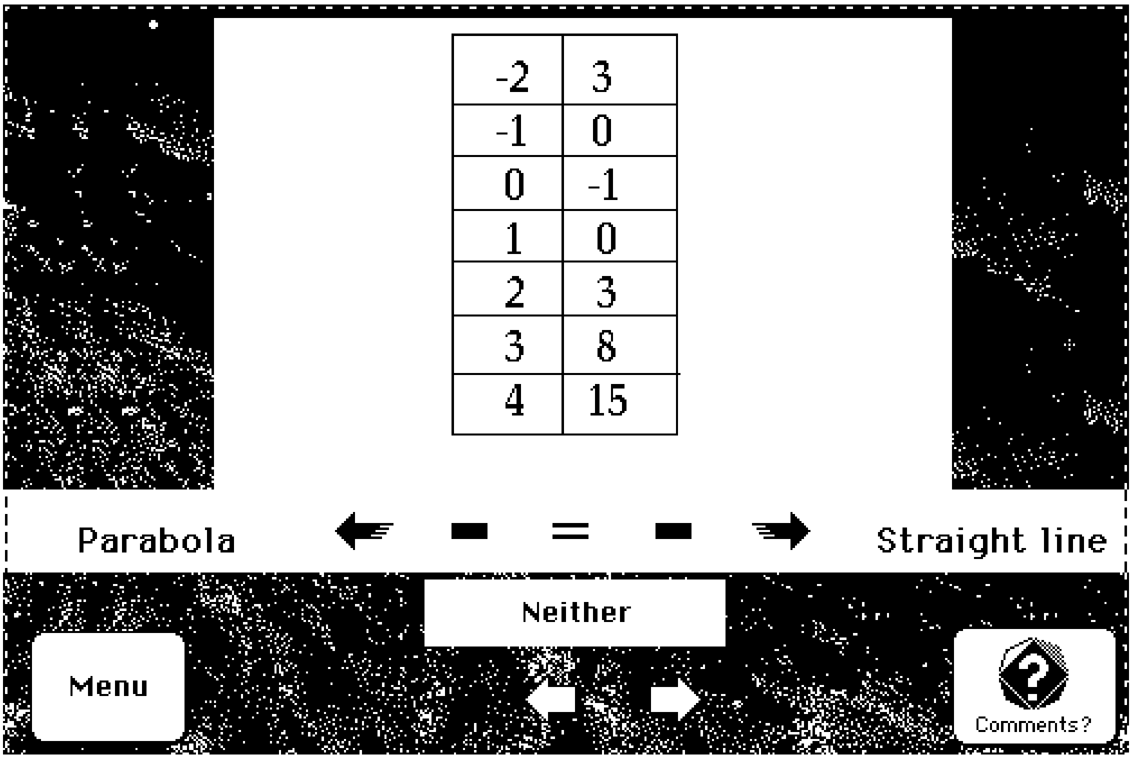
Later views of algebra are likely to incorporate a more active element as the previously representative object becomes a process of solving and finding answers. The use of additional representations, even among older students, was not common; of those mentioned, only the graphical form was offered as an alternative and, for some students, this did figure quite strongly in their thinking about algebra. Finally, algebra is seen to have no function beyond its own borders: its place in school and in mathematics is justified only by its own nature: algebra is studied in order to study more algebra. Such a view offers little motivation for its study for those for whom schooling is insufficient as an end in itself.

Understanding of algebra and its related concepts as demonstrated by verbal definitions appears to be quite fragile among all secondary students (and even some of the tertiary students), largely composed of disparate and poorly connected concepts. The need to expose the links between these ideas and the visual nature of many responses to algebraic concepts suggest that a study of algebraic imagery may reveal much more concerning the cognitive structures related to thinking about algebra.

Images of Algebra

What are termed here “images of algebra” were elicited through presentation of a series of ten cards, displaying a range of common algebraic visual prompts which participants responded to in a variety of ways. Participants were first asked to verbally describe each card, and then to sort them into as many groupings as they could (this may be considered a **first order grouping**). Four of the six Group A preservice teachers engaged in this process, as did all the secondary students. The secondary students then engaged in a more detailed discriminatory exercise, in which the ten images were presented three at a time, and students were asked to "choose the odd one out" - to decide in what way one image was different to the other two. This **second order grouping** exercise forced students to compare and contrast properties of the different images, and so potentially engage in a deeper analysis than the previous sorts.

Figure 6.2: Sample of a *RepGrid* analysis



As a final, in-depth analytical mechanism, Stephen, Tony, Patrick and the researcher (SMA/T1) engaged in a **third order grouping**, a detailed Repertory Grid analysis of the categories which arose from the previous discriminatory activity. Categories identified from the second order grouping were taken in pairs, placed as the end points of a continuum, and then presented with each of the ten original images. For example, I had distinguished between “parabola” and “straight line” in my second-order grouping. I was then asked to decide the extent to which each of the ten card images displayed these two properties by clicking at points between them (Figure 6.2). This process attempts to explicitly expose the network of relationships perceived by each individual in their thinking about algebra.

Images were chosen so as to offer the basis for sorts based both upon surface properties (for example, symbol/graph/numbers) and a range of possible other categories, such as functions/non-functions, different representations of the same symbolic form (cards 7 and 8) and even potential errors, such as equating the graph in card 2 with the visually similar symbolic forms of cards 7 and 8.

Before detailing the responses for each grouping, the initial verbal descriptions for each image are examined. These proved revealing of the level and nature of individual thinking regarding algebraic concepts, especially in comparison with the verbal definitions already described.

Card 1(expression): 4 - 3x

It has been suggested that one likely impact of computer technology upon the teaching and learning of algebra will be to move the central object of focus for the algebra curriculum from equation to function (Fey and Good, 1985, Kaput, 1992). Certainly the majority of computer tools used in algebra learning take the function as their principal object of action. An expression such as **4 - 3x** may be viewed in several different ways which are relevant in this context. In fact, different individuals were found to read a surprising variety of signals into such a simple object.

Student teacher A2, for example, observed first that this was simply "not an equation as such" (suggesting the dominance of the equation as an algebraic object of focus). Later, she expanded upon this:

> This is a number statement that could show as the value of x increases, the answer will be getting smaller. As x decreases (gets closer to zero) the answer will be getting smaller but will not be less than zero. If the x value goes below zero then the answer will be getting larger.

Note both the level of generality of the response (unrelated to the actual visual features, such as the 4 and -3x components) and the active process orientation, in which the value of the variable "x" is dynamically linked to the "answer". A2 does not say that as the value of x changes, the answer "gets" smaller; her implication in using "getting" seems to be of a highly fluid, dynamic relationship. In SOLO terms, this response might be classified as indicative of *formal operational thinking*, comparable to van Hiele's *Theoretical level*.

The response of A3 was more typical of multistructural concrete-symbolic thinking associated with the algebraic object: "From the numeral '4', a value calculated as 3 times another number 'x', is subtracted. Any number can be substituted for x, and thus the equation has multiple values". Once again, the response carries with it an active perception of process, this time related to the sequence of operations implied by the algebraic expression and also associated with the "function machine" image, of numbers "going in" and other numbers "coming out". Note the assumption by both individuals that the unspecified expression can act as an "implied equation".

For preservice teacher, A4, the immediate response ("This doesn't mean jack shit to me") was replaced by a more reasoned one: "This is a function; when x is greater than 1 1/3 our answer is negative, when x is 1 1/3 our answer is 0, when x is less than 1 1/3 our answer is positive". He then goes on to comment: "I would feel more comfortable if the card read 4 - 3x =". Like A2, this individual looks beyond the visual properties to respond almost automatically using equation-solving techniques. The equation signal is so strong that the expression is viewed as incomplete without an equals sign.

Both A5 and A6 responded in the same manner as did A3, reacting to the surface stimuli of the expression and responding to the operational process implied: "To me this is a statement which involves an unknown quantity. It represents three lots of something being subtracted from 4" (A5) and "This means 4 minus three times the value of an unknown number. The x can take any value unless some restriction has been placed upon it. In this form the expression can not be simplified. It is simply a generalisation of the idea expressed in my first sentence".

The responses of the preservice teachers fall clearly into two groups - those who responded to the immediate visual signal of the object, and those who operated at a higher level of generality. In all cases, however, there appeared to be a level of confusion regarding the signal nature of the expression: all wanted to *do* something to it - to substitute, to simplify or, in the majority of cases, to solve. These are the responses provided by traditional training in algebra - the desire for closure may be satisfied by action upon the algebraic object, and only three strategies are available, even to these highly trained students of mathematics.

The responses of the students to the expression **4 - 3x** were similarly revealing. While Patrick admitted readily that "this doesn't really mean much to me because I haven't worked with any of this stuff yet," Tony observed that it was, to him, "an equation and undoable question". Notice once again the desire to act stimulated by the algebraic expression, a desire frustrated by a lack of available strategies by which it may be operated upon. Jane and Ben both reacted to the visual stimulus - "taking 3x away from 4" (Jane) and "a simple subtraction equation where x is unknown" (Ben). The more experienced students, however, vented their desire to act by relating the expression to the coordinate plane: "line, y intercept at 4, gradient 3" (Andrea) and "this equation makes me think of a straight line with a gradient of 3 and crosses the x-axis at 1 1/3" (Stephen). The readiness of these students to assume a graphical metaphor by which to conceptualise the expression appears likely to result directly from their increased exposure to the graphical representation within their technology rich learning environment. Notice that, for Stephen at least, this graphical metaphor actually subsumed and included the equation-solving view of the preservice teachers (since he had used these techniques to arrive at the value of the x-intercept).

I had seen it as all these things: "An expression which describes a numerical process, acting upon an infinite array of numbers to produce a new infinite array. Graphically, this represents a straight line with negative slope (of -3) passing through 4 on the y axis". This relational response links the various elements of the previous multistructural responses.

Overall, then, responses appeared to fall into three levels of increasing complexity:

(1) an immediate, visual response, reacting to the *symbol* nature of the expression;

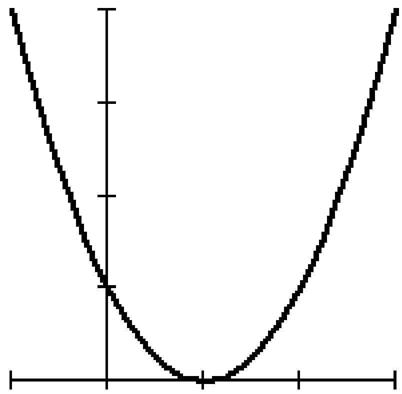
(2) a first-level signal response which views the expression as an equation to be acted upon, and

(3) a second-level signal response in which the object is viewed in both a cross-representational sense and as an "implied equation".

Card 2 (equation - *cf.* Card 9): $y = 2x - 1$

Participants were far more comfortable with this image than with the previous one. In every case (other than Tony - who saw it as an equation to be solved - and Patrick, who had insufficient algebraic experience to recognise it), the equation triggered the same response: as representing a graph with gradient 2 and y-intercept -1. The signal character of the equation in this form was strong and immediate.

Card 3 (parabola graph):



(Note that the symbolic form for this graph is $\mathbf{y = (x - 1)^2}$. This may be contrasted with both Cards 7 and 8, which offer different representations of a similar but distinct symbolic form.)

As with the equation, $\mathbf{y = 2x - 1}$, the responses were uniform and predictable. All recognised the signal of the parabolic form.

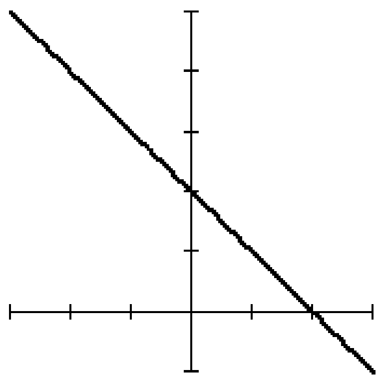
Card 4 (x, y pair): (x, y)

In all cases, this symbol was recognised as representing a point or position on the number plane. The signal was clear and unambiguous.

Card 5 (function symbol): $f(x)$

More varied were the responses to this symbol, ranging from the unistructural *"This is function x"* (A4) and *"This represents a function with respect to x"* (A5) to identification with numerical values (A3, A6, Andrea and Stephen), pairs of numbers (Ben) and points on a graph (Jane). The symbol appeared to trigger either an active numerical image or a more static graphical one.

Card 6 (graph of y = 2 - x)



Once again, a uniform response from all participants. All noted that it was a straight line on the number plane, and most identified gradient as -1 and y intercept as 2.

Card 7 (Table: $y = x^2 - 1$)
(Note that the factored form of this function is $(x - 1)(x + 1)$: *cf*. Card 8 and contrast with Card 3).

-2	3
-1	0
0	-1
1	0
2	3
3	8
4	15

Individuals responded in three distinct ways: focusing upon the table as points for a graph (Jane, Tony and Patrick), as input/output pairs (Stephen) or as the result of a rule or equation (A5, A6, Andrea and Ben).

Card 8 (expansion)
(*cf*. Card 7, contrast with Card 3)

$$(x - 1)(x + 1)$$

Responses to this card provide examples of what van Hiele refers to as a **rigid structure** - the signal to act by expanding to give the familiar "difference of two squares" was almost overpowering. All responded algebraically, while only Andrea, Ben and Stephen reacted graphically as well. Once again, the student teachers appear to react differently to the students who have had exposure to computer tools and, as a result, have become comfortable with the graphical representation.

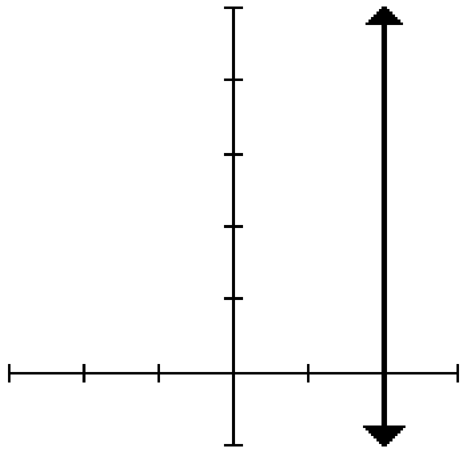
Card 9 (equation)
(*cf*, Card 2)

$$2x - 1 = x + 7$$

Like the expansion of card 8, participants appeared to react positively to this prompt, apparently feeling themselves to be once more upon familiar ground. All responded to the manipulative signal to act which had been so effectively instilled in them by their algebraic instruction. Interestingly, Andrea, Ben, Jane and Stephen all indicated solutions involving the physical movement of parts of the equation “across the equals sign”.

Card 10 (graph of x = 2)

(Note that this is an example of a non-function). Participants saw it as a straight line, with most identifying it as x = 2. It triggered no other responses.



Analysis of the ten “images of algebra” across all participants and all grouping activities led to the identification of eleven principal **descriptors**, with several sub-categories related to the most common occurrences of some of these. These are displayed along with the participants from whom they were derived in Table 6.2.

Since all these participants responded to the same ten visual prompts to produce the descriptors listed, the table gives some indication of both the relative complexity of the thinking about algebra displayed by different individuals (viewing vertically) and the relative frequency of occurrence of the different descriptors (viewing horizontally).

Table 6.2

Algebra Descriptors

	A2	A3	A4	A5	A6	S1	S2	S3	S4	S5	S6	T1
1. Equation	√	√	√	√	√	√	√	√	√	√	√	√
1. linear	√		√	√	√	√	√	√	√	√		√
2. quadratic				√	√	√	√	√	√			√
3. Non-equation	√					√	√					√
2. Graph	√	√	√	√	√	√	√	√	√	√	√	√
1. linear	√	√		√	√	√	√	√	√	√		√
2. quadratic	√	√	√	√	√	√	√	√	√	√		√
3. non-graph						√	√		√	√	√	
3. Function	√	√	√	√	√	√	√	√	√			√
2. quadratic									√			
3. non-function							√		√			
4. Expression/symbol		√		√	√	√	√	√	√	√	√	√
5. Signal to action	√	√	√	√	√	√	√	√	√	√		√
6. Numerical value	√	√	√		√	√	√	√	√	√	√	√
7. Function machine	√				√		√	√	√			√
8. Relationship	√	√			√							√
9. Table of values	√	√		√	√		√	√	√	√		√
10. Variable	√	√	√	√	√		√		√	√		√
11. (x,y) Coordinates	√	√	√	√	√	√	√	√	√	√	√	√
- Non-coordinates						√			√	√	√	√

With regard to the individual participants, it is not surprising to find S6 (Patrick) displaying the most limited range of descriptors, along with S5 (Tony) and S3 (Jane). This offers validation to the findings based upon these student's verbal definitions, to suggest that their network of algebraic concepts is less well developed in comparison with the older participants. Even among the preservice teachers, differences in complexity are apparent, with A4 displaying the most limited repertoire of ways of thinking about algebra. Neither is it surprising to find the most extensive range of descriptors associated with myself as teacher/researcher, both because of my greater mathematical experience, and also because of my role in designing the research questions and analysing the results.

Most frequent descriptors across both students and preservice teachers were the common algebraic forms - equations, graphs and (x, y) coordinates. Distinctions between graphs - straight lines and parabolas - were also evident for all but Patrick (S6), and there was general recognition of functions, variables and expressions, although these terms were not always used appropriately (*equation* and *expression*, for example, tended to be interchangeable terms). Common, too, was an active conception of algebraic forms - the signal characteristic discussed previously. The identification of participant categories, however, was only a first step in the research design. The final aim was to make explicit relationships which might exist between these categories, and to expose the nature of the cognitive network by which the various individuals conceptualise algebra.

First Order Groupings

A useful measure of the complexity of individual thinking about algebra was offered by the first order groupings of the image cards. This process had an immediacy which tended to be revealing of the *signal nature* of the various algebraic forms. Although at times the groupings were idiosyncratic, there were clear patterns of consistency which went beyond the "surface" characteristics of the cards.

My own groupings, both as an experienced mathematics educator and as research designer, were, not surprisingly, the most comprehensive.

Six groupings were identified which provide perhaps as much insight into my perceived priorities within the research design as into my algebraic thinking.

Table 6.3

First order groupings: SMA/T1

Group 1: *Expressions*	**Group 2:** *Equations*	**Group 3:** *General forms*	**Group 4:** *Linear graphs*	**Group 5:** *Parabolas*	**Group 6:** *Pairs of numbers*	**Group 7:** *Non-functions*
4 - 3x (x-1)(x+1) *f*(x)	2x-1= x+7 y = 2x - 1	(x, y) *f*(x)	Gr (y=2-x) Gr (x=2) y = 2x-1 2x-1=x+7 4 - 3x	Table Gr (parab.) (x-1)(x+1)	Table (x, y) Gr (y=2-x) Gr (parab) y = 2x-1 Gr (x=2)	Gr (x=2)

The explicit recognition of non-functions, the association of graphs, table and equations as representing "pairs of points" and the inclusion of the equation **2x-1 = x+7** and the expression **4 - 3x** with other "linear graphs" signify a perspective quite distinct from that of the other participants. This difference will influence both the research design and the analysis which follows.

Of the preservice teachers, A2 produced only two groups from her single sort. These groups were mutually exclusive - all cards were included once except the *(x, y) pair* which was apparently overlooked.

Table 6.4

First order groupings: A2

Group 1: *Showing y as a function of x*	**Group 2:** *shows values of x*
y = 2x-1 *f*(x) Table of values Graph y = 2-x Graph parabola Graph x = 2	4 - 3x (x - 1) (x + 1) 2x - 1 = x + 7

A4 proposed four groups, reusing several prompts in the process.

Table 6.5

First order groupings: A4

Group 1: *Graphs*	**Group 2:** *Functions of y*	**Group 3:** *Coordinate Geometry*	**Group 4:** *Solving equations*
Graph (y = 2- x) Graph (parabola) Table of values Graph (x = 2) y = 2x - 1	(x, y) Graph (parabola) Graph (y = 2 - x) Graph (x = 2) Table of values	(x - 1)(x + 1) y = 2x - 1 2x - 1 = x + 7	4 - 3x (x - 1)(x + 1)

Group 1 suggests that, like Stephen, A4's thinking about algebraic objects is strongly influenced by their visual format. While the function symbol, the linear equation and even the graphs and table of values were seen as explicitly denoting a functional form, the two expressions and the equation to be solved were not. Rather, these were seen as representing values of x (and, by implication, *not* values of y). An expression such as **4 - 3x** appears to lack the signal character of an equation such as **y = 2x - 1**. Note, too, the incorrect inclusion of the graph x = 2 as a function.

It should be noted here that Groups 1 and 2 are essentially the same, with only the equation **y = 2x - 1** and the ordered pair **(x, y)** to distinguish them (even these appeared arbitrary, since the first could certainly be considered a "function of y" and the ordered pair is commonly associated with "graphs", consistent with A4's association of the table of values with the cards indicating graphs.) Again, the graph of the deliberate non-function, **x = 2**, was grouped with "functions", and the two **expressions** were incorrectly labelled as **equations** (Group 4). This last suggests that A4 may see an implied equation form, such as **4 - 3x = 0** when viewing the expressions. This grouping excluded the function symbol, ***f*(x)** and, overall, tended to suggest a poorly organised cognitive network related to algebra.

A5 and A6 showed surprising consistency in their cognitive organisation. Both created five categories, which included straight lines, parabolas, equations and graphs. Both recognised the **table of values** and the expression **(x - 1)(x + 1)** as representations of quadratic forms, suggesting an ability to move across symbolic, graphical and numerical representations, and indicative of a relatively high level of functioning in this domain. At the same time, neither clearly distinguished expressions from equations, nor explicitly differentiated functions from non-functions, even though all preservice teachers had included the uniqueness property within their verbal definitions of function. In fact, A2, A4 and A6 incorrectly placed the graph **x = 2** within groupings designated as "functions". Knowledge of the formal definition does not necessarily imply the ability to correctly apply it, a result which echoes the findings of Vinner and Dreyfuss (1989).

Table 6.6

First order groupings: A5

Group 1: *Straight lines*	**Group 2:** *Parabolas*	**Group 3:** *Equations*	**Group 4:** *Statements*	**Group 5:** *Graphs*
Graph (y= 2-x) y = 2x - 1 Graph (x = 2)	Graph (parab.) Table of values (x-1)(x+1)	y = 2x - 1 Graph (parab.) 2x-1 = x +7 Graph (x=2) Graph (y=2-x) (x-1)(x+1)	4 - 3x y = 2x - 1 2x - 1 = x + 7 (x - 1)(x + 1)	Graph (parab.) Graph (y = 2-x) Graph (x = 2)

Table 6.7

First order groupings: A6

Group 1: *Linear equations*	**Group 2:** *Quadratic equations*	**Group 3:** *Functions*	**Group 4:** *Equations*	**Group 5:** *Coordinates/ Graphs*
4 - 3x Graph (y= 2-x) Graph (x = 2) y = 2x - 1	Table of values Graph (parab.) (x-1)(x+1)	Table of values *f*(x) Graph (parab.) Graph (x=2) Graph (y=2-x)	y = 2x - 1 2x - 1 = x + 7	(x, y) Graph (y = 2-x) Graph (parab.) Table of values Graph (x = 2)

The groupings produced by the secondary students were similarly diverse. Andrea (S1) identified three categories only, but these spanned the symbolic and graphical representations. Thus she correctly grouped both graphical forms of parabola and straight lines with their symbolic forms, and included the expression **4 - 3x** and the equation **2x-1 = x+7** as straight lines. She was not, however, able to interpret the table of values in order to recognise its quadratic nature, nor was she able to place the ***f*(x)** symbol in relation to the other cards. For a student attempting the high level Three Unit course in mathematics, this grouping suggests a limited cognitive repertoire regarding algebraic understanding, but one within which the graphical representation plays a significant part.

Table 6.8

First order groupings: S1 (Andrea)

Group 1: *x & y coordinates*	**Group 2:** *Parabolas*	**Group 3:** *Straight lines*
(x, y) Table of values	Graph (parabola) (x - 1)(x + 1)	Graph (y=2-x) 4 - 3x y = 2x-1 Graph (x=2) 2x-1 = x+7

Like Andrea, Ben’s sort appeared to be strongly influenced by surface characteristics of the algebraic images presented, although Andrea’s categories related to algebraic properties, while Ben's arose from visual cues. There was little evidence of insight into connections between the various cards beyond that suggested by their visual appearance. As with most other participants, equations and expressions are considered interchangeable terms and, in Ben's case, the ***f*(x)** symbol is associated with graphs.

Table 6.9

First order groupings: S2 (Ben)

Group 1: *Expressions of algebra*	**Group 2:** *Graphs*	**Group 3:** *Pairs*	**Group 4:** *Equations which can be solved*
(x-1)(x+1) 4 - 3x 2x-1 = x+7 y = 2x-1	Graph (y = 2-x) Graph (parabola) Graph (x = 2) *f*(x)	Table of values (x, y)	2x - 1 = x + 7 y = 2x-1 (x-1)(x+1)

Jane engaged twice in the algebra card sort, offering four groups initially and seven groups later in the research process. She chose to use both graph plotter and table of values during the second sort to examine two of the images more closely (the graphical image of the line y = 2 - x, for which Jane used the graph plotter to ascertain the values of the intercepts, and the table of values card - after entering the values into the table of values utility, she then plotted these using the graph plotter). This use of tools supported increasing breadth and depth of analysis as suggested by the Vygotskian **Zone of Proximal Development**. Jane used the available tools to go beyond her current cognitive state and consequently recognised properties of the algebraic images which, while meaningful, were beyond that which she could have done unaided (she recognised, for example, the quadratic nature of the table of values image).

Table 6.10

First order groupings: S3 (Jane): Sort 1

Group 1: *Algebra*	**Group 2:** *Graphs*	**Group 3:** *Functions*	**Group 4:** *Equations*
4 - 3x (x, y) (x-1)(x+1) 2x-1 = x+7 y = 2x-1 *f*(x)	Graph (y = 2-x) Graph (parabola) Graph (x = 2) Table of values	(x, y) *f*(x) y = 2x-1 Table of values	2x - 1 = x + 7 y = 2x-1

Jane's first sort is surprisingly similar to that of Ben, both offering a "grab-bag" of algebraic forms in Group 1 (possibly anything with an "x" in it), and very similar groups for **graphs** and **equations**. Jane's use of the term **functions** as a category is interesting in the light of her uncertainty regarding its meaning which had been exposed in the earlier interview. This had perhaps sensitised her to the term and led to an increased awareness of its occurrence. Her choices for this group were all appropriate, suggesting that she is able to recognise examples of its occurrence to at least a limited extent.

Jane's second sort suggests increased cross-representational facility - the table of values is recognised as representative of both graph and parabola and the symbolic forms, **y = 2x - 1, (x, y)** and ***f*(x)** are readily associated with graphs as well as input/output numbers. It would appear that these algebraic images have moved through the use of software tools and from exposure to a technology-rich learning environment from possessing a **symbol nature** as demonstrated in sort 1 to acting as meaningful cognitive **signals** in the later sort.

Table 6.11

First order groupings: S3 (Jane): Sort 2

Group 1: *Graphs*	**Group 2:** *Algebra*	**Group 3:** *Equations*	**Group 4:** *Involve numbers*	**Group 5:** *Substitution*	**Group 6:** *Straight line graphs*	**Group 7:** *Points on a graph*	**Group 8:** *Parabolas*
Gr (y=2-x) Gr (par.) Gr (x=2) (x, y) Table *f*(x) y = 2x-1	(x-1)(x+1) 2x-1=x+7 y=2x-1 4 - 3x *f*(x)	2x-1 = x+7 y = 2x-1	4 - 3x Table (x-1)(x+1) y = 2x-1	*f*(x) (x, y)	Gr (y=2-x) Gr (x=2) y = 2x-1	Table (x, y) *f*(x)	Table (x-1)(x+1) Gr (par.)

Like Jane, Stephen (S4) also engaged twice in the algebra card sort activity and, also like Jane, increased over the intervening period from four categories to seven categories. He chose, however, not to make use of available software tools. His first sort was restricted in that he used each card only once, and so sorted them into exclusive categories. He displayed limited cross-representational facility, recognising the equation **y = 2x-1** as a linear graph and the expression **(x-1)(x+1)** as representing a parabola. He also treated the expression, **4 - 3x**, as an “implied equation”, capable of solution if “**= 0**” is assumed as a suffix. Functions were included only as symbolic and numerical forms (the table of values implying for Stephen an “input/output” image suggestive of function).

Table 6.12

First order groupings: S4 (Stephen): Sort 1

Group 1: *Functions*	**Group 2:** Solving equations	**Group 3:** *Straight lines*	**Group 4:** *Parabolas*
f(x) Table of values (x, y)	4 - 3x 2x-1 = x+7	y = 2x-1 Graph (y=2-x) Graph (x=2)	(x-1)(x+1) Graph (parabola)

Table 6.13

First order groupings: S4 (Stephen): Sort 2

Group 1: *Function*	**Group 2:** *Straight line*	**Group 3:** *Parabola*	**Group 4:** *Equation*	**Group 5:** *Coordinates*	**Group 6:** *Equation for axis*	Group 7: *Find values for variables*
Table *f*(x)	Gr (y=2-x) Gr (x=2) y = 2x-1	Gr (parab.) (x-1)(x+1)	4 - 3x 2x-1 = x+7 y = 2x-1 (x-1)(x+1)	(x, y) Table	Gr (x=2) y = 2x-1 Gr (parab.) Gr (y=2-x)	4 - 3x (x-1)(x+1) 2x-1 = x+7

Although Stephen demonstrated the most versatile and deep understanding of algebraic concepts of the student group, his second sort showed less improvement than did Jane's. His increased number of groupings appeared generally consistent but somewhat arbitrary (as in "Equation for axis"), and overall this sort demonstrated little improvement in his cognitive organisation than that which was made evident in the first. Although he showed good familiarity with the graphical representation, he appeared unable to interpret the table of values in a meaningful way.

Tony engaged in three first order sorts over a period of twelve months. Although the number of groupings increased in that period (from three to five), they remained based firmly upon superficial features of the images.

Table 6.14

First order groupings: S5 (Tony): Sort 1

Group 1: *Equations*	**Group 2:** *Number planes*	**Group 3:** *Things I don't understand*
(x - 1)(x + 1) 2x-1 = x+7 y = 2x-1	Graph (parabola) Table of values Graph (y = 2-x) (x, y) Graph (x=2)	4 - 3x *f*(x)

Interestingly, Tony's second sort reduced the number of groups to two:

Table 6.15

First order groupings: S5 (Tony): Sort 2

Group 1: *Graphs*	**Group 2:** *Equations and algebra*
Graph (y = 2-x)	4 - 3x
Table of Values	(x - 1)(x + 1)
Graph (parabola)	y = 2x - 1
(x, y)	2x - 1 = x + 7
Graph (x = 2)	*f*(x)

This second sort demonstrated improvement in both the appropriate use of technical terms ("graphs" instead of "number planes") and a clear distinction between what, for Tony, are the two fundamental divisions within algebra: symbols and graphs.

Tony's third sort displayed a finer detail and a better grasp of the language of algebra ("coordinates" and correct use of the term "expression") but not necessarily a deeper understanding of the distinctions between the various images. He now has a name for those "things I don't understand" from sort 1, and the expression **4 - 3x** has acquired the signal property of "something to be solved" which leads to inclusion with the equations - although, once again, the other expression, **(x-1)(x+1)**, was omitted (it had been seen as an "equation" in the first sort). Tony had studied expansion of binomials at school by this time, and indicated that he "knew what to do with this one", implying that he saw **(x-1)(x+1)** as something to be expanded. It seems possible that the stronger "expansion" signal served to "swamp" the "equation" signal in this case. Note that he correctly places the symbol ***f*(x)** as an expression, showing recognition of this form.

Table 6.16

First order groupings: S5 (Tony): Sort 3

Group 1: *Graph*	**Group 2:** *Equations*	**Group 3:** *Coordinates*	**Group 4:** *Table*	**Group 5:** *Expression*
Gr (parabola) Graph (y=2-x) Graph (x=2) (x, y)	y = 2x-1 2x-1 = x+7 4 - 3x	(x, y) Table of values	Table of values	4 - 3x (x-1)(x+1) *f*(x)

Finally, as a novice to algebra, Patrick's sorts would be expected to be based upon superficial cues, since the images possess for him no underlying meaning. His groupings reflect this.

Table 6.17

First order groupings: S6 (Patrick): Sort 1

Group 1: *Graphical*	**Group 2:** *Numbers and letters*	**Group 3:** *Mix*	**Group 4:** *Loner*	**Group 5:** *Problems*
Gr (parabola) Gr (y=2-x) Gr (x=2)	4 - 3x (x, y) (x-1)(x+1) 2x-1=x+7 y = 2x-1 *f*(x) Table of values	4 - 3x Gr (y=2-x) Table of values (x, y) Gr (parabola) y = 2x-1 *f*(x) Gr (x=2) 2x-1 = x+7 (x-1)(x+1)	*f*(x)	(x, y) 4 - 3x y = 2x-1 2x - 1 = x + 7 (x-1)(x+1)

Even at this early stage, Patrick recognised that algebraic forms possess a signal nature: his last group, "Problems", indicates objects which he saw as requiring some action, although he was unsure of what that action should be. His second sort was much more clearly defined than the first, with three clear and distinct groupings which again reflect the surface features of the algebraic forms.

Table 6.18

First order groupings: S6 (Patrick): Sort 2

Group 1: *Graphs*	**Group 2:** *Equals*	**Group 3:** *Numerals*
Graph (y = 2-x)	(x-1)(x+!)	4 - 3x
Graph (parabola)	2x-1 = x+7	(x, y)
Graph (x=2)	y = 2x-1	Table of values
		f(x)

Patrick's inclusion of the expression **(x-1)(x+1)** within the group labelled "equals" and the presence of the symbolic forms **(x, y)** and ***f*(x)** (and even the expression **4 - 3x**) among "numerals" reveals his recognition of the implicit and symbolic nature of algebra, where the symbols possess meaning beyond their surface appearance. Such a recognition appears to signify an important step forward in algebraic understanding. This understanding is probed further through the use of deliberate comparative techniques which give rise to the next level of image sorts.

Second Order Groupings

The deliberate comparing and contrasting of algebraic images (picking the "odd one out") offers an added degree of depth to the analysis of participant responses, forcing them to go beyond the often-superficial viewing associated with a verbal description. In particular, respondents who had difficulty in supplying verbal descriptions are provided with a non-verbal means of conveying elements of their thinking about algebra. At the same time, these non-verbal responses are supplemented by comments regarding the choice made, which provide further insight into the reasons for these choices (text records for these comments are recorded in Appendix E). Each of the student participants (except Jane) provided responses to this activity (Jane had left the program before this instrument had been developed); my own responses (SMA) are included as a further bracketing device (Table 6.19)

Table 6.19

Second Order Groupings: "*Pick the odd one out*"

		Andrea	Ben	Stephen	Tony	Patrick	SMA
1	4-3x y = 2x-1 Gr (y=2-x)	**1**	**3**	**1**	**1**	**3**	**2**
2	Gr (par) (x-1)(x+1) Table	**1**	**1**	**3**	**2**	**1**	**1**
3	(x, y) 2x-1=x+7 Table	**2**	**3**	**2**	**2**	**3**	**2**
4	*f*(x) 4 - 3x Gr (x=2)	**2**	**2**	**2**	**2**	**3**	**3**
5	(x-1)(x+1) 2x-1=x+7 4 - 3x	**1**	**1**	**1**	**1**	**3**	**2**
6	Gr (y=2-x) Gr (par) Gr (x=2)	**2**	**2**	**2**	**2**	**3**	**3**
7	y = 2x-1 (x, y) 2x-1 = x+7	**2**	**2**	**3**	**2**	**2**	**1**
8	y = 2x-1 *f*(x) 2x-1 = x+7	**2**	**2**	**3**	**2**	**2**	**1**
9	Gr (y=2-x) Gr (par) Table	**1**	**2**	**3**	**1**	**3**	**1**
10	*f*(x) (x, y) (x-1)(x+1)	**3**	**1**	**3**	**3**	**3**	**3**

The sorting of the ten images into triads was, in most cases, deliberate rather than random. Triad 1 grouped the three major algebraic categories: expression, equation and graph. Ben and Patrick both used an immediate visual distinction, and I distinguished on the basis of negative gradient. The remaining students associated **y = 2x-1** with the graph and saw the expression as the “odd one out”.

Triad 2 grouped graph, table of values and expression, with mixed responses. The direct correspondence between the rule for the table of values and the algebraic form was recognised only by myself and Andrea, demonstrating a high level of cross-representational facility on her part. She had learned to interpret the table as a representational form. Ben also nominated the graph as the “odd one out”, but for the superficial reason that both table and expression had “x values of -1 and 1”. Stephen recognised that both graph and table represented parabolas, while Tony equated the tabular form with the graphical form. This item exposed a distinct hierarchy of responses:

Level 0: Patrick and Ben, who responded at a purely visual level,

Level 1: Tony saw the number pairs of the table as a general signal to graph these points.

Level 2: Stephen transferred meaning across the graphical and tabular representations, but on a visual level only.

Level 3: Andrea transferred meaning on a symbolic level.

The third triad demonstrated that, for most participants, the link between the tabular representation and the ordered pair symbol, **(x, y)**, was a strong one. Although, once again, Ben and Patrick responded in the same way, they did so for different reasons. Patrick focused upon the superficial “equals” prompt, which he saw as implied in the ordered pair but not in the table of values. Ben, on the other hand, contrasted the “known x and y values” of the table with the unknowns of the other two forms.

Triad 4 exposed the strong link between the ***f*(x)** symbol and the graphical representation for all students but Patrick. Patrick distinguished the graph from the “non-graphs”, while I distinguished functions from non-function.

Triad 5 was similarly consistent across the students, but for a range of reasons. Andrea and Ben responded to the graphical representation implied, distinguishing between quadratic and linear forms. Tony responded to the “expansion” signal of the expression, while I focused upon expressions as opposed to equations. Stephen’s verbal response to this item sees him again view the expression **4-3x** as an “implied equation”, while **(x-1)(x+1)** does not have this property: “For the two that are alike you find an x-value, and with the other one it just describes a function”. Clearly, for Stephen, the expression **(x-1)(x+1)** possesses a much more limited signal character than the linear expression.

It is hardly surprising that item 6 produced such a consistent response - the visual signal of curved/straight, parabolic/linear produced a prompt reaction from all students but Patrick, who saw the vertical line as somehow different to the other two (but could not say why), while I focused upon the function/non-function distinction.

A similar strong visual signal was sent by the equals sign in both Triads 7 and 8. Stephen’s response to both items, however, was strongly influenced by the graphical representation which he brought to the question, linking **y=2x-1** and **(x, y)** in Triad 7 ("The similar ones are describing a straight line with two pronumerals (x and y), and the other one just defines an x-value as a number not in terms of y"), and **y=2x-1** and the symbol ***f*(x)** in Triad 8 ("The similar ones are functions and describe a graph or what you get if you put x-values in and make a picture, and the odd one out just finds the x-values"). Once again, Stephen's strong reliance upon the function-machine metaphor for thinking about functions is evident here.

Item 9 caused a range of responses: Andrea and Tony both demonstrated again that they are able to interpret the table of values representation, distinguishing parabolas from straight lines; Stephen and Patrick both responded superficially upon the basis of graphs and table forms, while Ben chose the parabola as the odd one out since it does not go below the x-axis, whereas the other two do. This demonstrates a limited ability to interpret tabular information.

Ben was also the "odd one out" in his response to the last item, seeing ***f*(x)** as a more general symbolic form than **(x, y)**: "The other two could both be just points, while *f*(x) could be anything". The other respondents accorded both symbols equal levels of generality, as opposed to the more specific expression.

The overall impression gained from these second order groupings is that of students who are all able to demonstrate quite strong cross-representational facility, at least across symbolic and graphical forms. Interpretation of the tabular form is much more limited, consistently found only in Andrea and apparently all but absent in Stephen, who prefers to think graphically. Along with Tony and Patrick, Ben displays a very limited representational repertoire.

Third Order Groupings

The task which gave rise to the third order groupings was a very time-consuming one, taking up to thirty minutes to complete. For this reason, only four participants were engaged in this activity - myself, Stephen (as the principal informant) and the two junior secondary students, Tony and Patrick (whose limited algebraic experience meant that they had been restricted in their access to appropriate language and forms of expression by which their understanding might be examined). The previous tasks in these two cases had furnished limited information regarding their algebraic thinking - it was hoped that this detailed analysis might provide a useful non-verbal vehicle by which their cognitive frameworks might be better assessed.

This task involved three steps:

1. The verbal statements which had accompanied the second order grouping process were examined, and used to give rise to a number of descriptors which appeared to figure prominently in their thinking about the algebraic images. This process of extraction took place in collaboration with the informant, increasing validity for the descriptors.
2. The descriptors were entered into the *HyperCard* RepGrid stack and participants would again view each card individually. This time, however, instead of requiring a verbal descriptor, the descriptors would be displayed in pairs, as the ends of a continuum (see Figure 6.2).
3. Participants would choose an appropriate response which situated the given image in relation to the two descriptors.

In my case, the ten comments which accompanied my second order grouping process (Table 6.19) were:

1. They both have negative gradient.
2. They both represent $x^2 - 1$, while this one is $(x-1)^2$.
3. They imply an infinite set (or at least multiple ordered pairs); the equation implies a single pair.
4. This is a non-function; the others are functions.
5. The others are expressions (implying an infinite array of values); this is an equation.
6. Non-function.
7. The ordered pair representation for the others is explicit in x and y; for the equation it is implicit.
8. The other two each represent individual functions; the equation represents the equality of two distinct functions.
9. The others both represent parabolas.
10. Both the others are purely symbolic forms.

These comments gave rise to seven descriptors:

- function
- non-function
- equation
- expression
- graph
- table of values
- symbols

Stephen’s comments for each of the ten triads were as follows:

1. The other two both have two pronumerals (x and y) but 4-3x has only one. It doesn't tell you what comes out if you put numbers in.
2. The other two are both parabolas and the other one is just a set of values which you put in and something comes out.
3. The other two both have one value going in and another coming out, but in the third one it is just finding the x-value.
4. The others are a function with special values - whatever you put in you get different answers, but in the third one it’s just a statement - it doesn't show what you get out.
5. For the two that are alike you find an x-value, and with the other one it just describes a function.
6. The two alike are just straight lines, and the odd one out is a parabola.
7. The similar ones are describing a straight line with two pronumerals (x and y), and the other just defines an x-value as a number not in terms of y.

8. The similar ones are functions and describe a graph or what you get if you put x-values in and make a picture, and the odd one out just finds the x-values.
9. The other two are graphs on x and y axes, and the odd one out is a table of values - you don't really know what it is describing, which side is x or y.
10. The odd one out is not equal to anything, and the similar ones are describing a function.

From these were derived seven descriptors which encapsulated what for Stephen appeared to be key concepts regarding algebra:

- two pronumerals
- parabola
- function
- straight line
- graph
- table of values
- equation

Tony's comments were:

1. Because one had an x and a y in it, and the other was a graph - they both use x and y coordinates.
2. Because the table represents a graph and that ones a graph.
3. Because x and y are coordinates and the table is used in representing graphs.
4. Because the f(x) one has something to do with graphs .
5. Because the odd one out involves expansion and the other two are simple equations.
6. Because they are straight and the other one is a parabola.
7. Because the other 2 are equations and the one in the middle is a coordinate.
8. Because the other 2 are equations (again) and the odd one is a coordinate.
9. Because the table plots a parabola out and the other one is a straight line
10. The top 2 represent something to do with graphs and the bottom one has something to do with expansion.

These led to the identification of eight descriptors:

- graph
- coordinates
- table
- x and y
- equations

- expansion
- parabola
- straight line

Finally, Patrick's comments are given:

1. the other 2 are numerals but this one was a picture
2. I'm not sure about this one so I'm skipping this one.
3. The other 2 are describing how one thing equals another.
4. This one is a graph but the other 2 aren't.
5. I don't know.
6. the other 2 are marked by certain coordinations but this one isn't.
7. The other 2 describe what eg. x=, y=.
8. This is a symbol the other 2 aren't.
9. the 2 matchies are picture graphs but this one isn't.
10. the others are symbols but this one isn't.

Once again, seven descriptors were identified as arising from these:

- numerals
- picture
- not sure
- graph
- equals
- coordinations
- symbol

The full text of the third order groupings for each individual is recorded in Appendix E.

Analysis of the responses for each image card allows quite detailed assessment of individual thinking in relation to that concept. Responses for each card will be considered for the four respondents.

For myself, the algebraic expression **4 - 3x** elicited responses which classified it as symbol, function, expression, graph and table of values, but not an equation or non-function. It was seen as possessing characteristics of both graph and table of values, although more symbol than either of these. For Stephen, however, it was an image which was largely devoid of meaning - it was not associated with any of the categories chosen. It had no graphical or table of values form, was not a function (since it did not possess the “y=“ prefix) and was not even associated with a straight line. Stephen’s response to this item not only clearly highlights his erroneous conception of function, but demonstrates that the expression as an algebraic form has little signal character.

The expression elicited a similarly negative response from Tony, although he did associate it with “equation” and, interestingly, with table of values but not graph. For Tony, the table of values appeared to be perceived as a more flexible representation than the graphical form, more closely related to both formulas and numerical values.

For Patrick, the expression was dominated by its association with numerals, coordinations (Patrick’s expression for the “x” and “y” coordinates of algebra) and its symbolic nature. It was not linked to graphs, pictures or “equals” (equation forms). He perceived the other expression, **(x-1)(x+1)**, in exactly the same way.

The symbolic **(x, y)** pair was perceived by me as representing equation, graph, table of values and symbols, but not function, non-function nor expression. The explicit “x” and “y” elements implied an association of equality which could be represented in various forms, but excluded the notion of expression which was considered to be defined by its *lack* of an explicit y-variable. Function and non-function links were rejected because the pair could equally represent either form.

For Stephen, the association of the ordered pair with two pronumerals was an obvious one, but it possessed partial characteristics of other responses. It was, for example, considered to be “more two pronumerals than straight line or equation”, and yet “more graph and table of values than two pronumerals”. It was seen to possess some elements of parabola, straight line and function, and equally both graph and table of values. This partial categorisation suggests a more mature and flexible cognitive network than those which are strictly “black and white”, in which each item is purely one thing or another (which was more evident among the two younger students). Stephen’s ability to perceive in algebraic images various “shades of gray” suggests that his thinking is relatively rich and diverse.

For Tony, the **(x, y)** pair was strictly “coordinates” and “x and y”, and possessed characteristics of both graph and table of values, but not of equation, expansion, straight line or parabola. For Patrick, the image was “coordinations”, “numeral” and “symbol”, with some lesser association with “graphs” and “equals”.

The responses to the three graphical images were quite consistent for all participants. In my own case, I distinguished between functions and non-functions, and noted that the images possessed an element of the table of values form. Tony, similarly, saw the graphs as “more graph than table of values”, while Stephen rated them equally - for Stephen, a graph was as much associated with the numerical values from which it was derived as with the graphical form. Patrick, too, recognised the role of "coordinations" as of equal importance in determining the graph as the visual form.

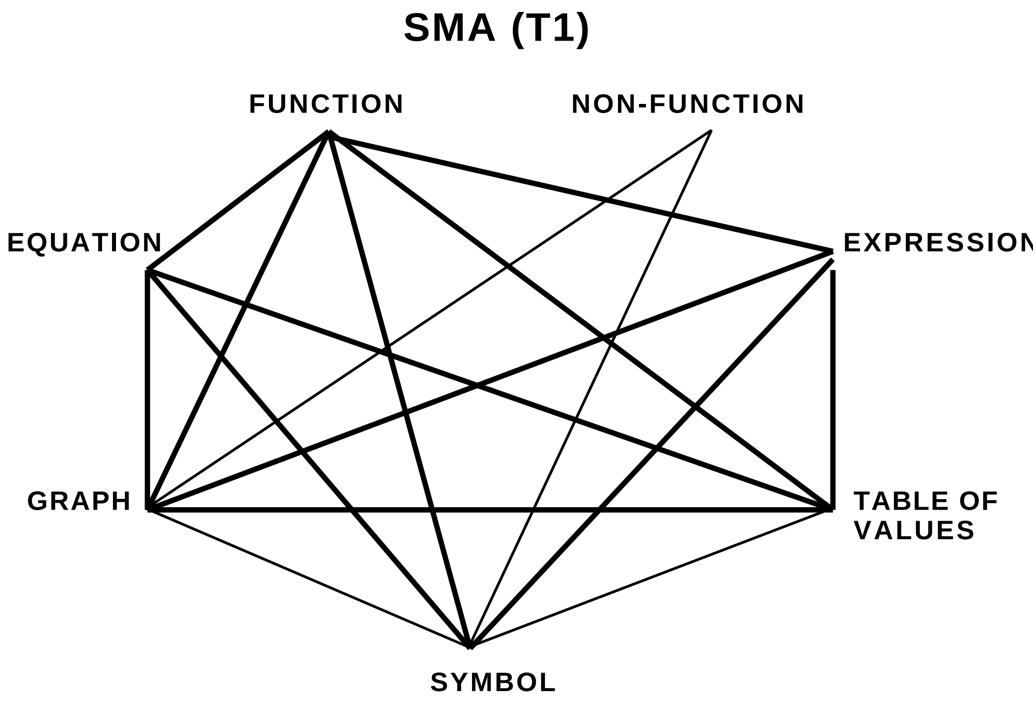
The ***f*****(x)** symbol drew some varied responses. While I associated it with function, expression, graph, table and symbol, I rejected association with equation, since the required "=" symbol is absent. Stephen, on the other hand, saw it as linked to everything, possessing elements of equation, function, graph, table of values, even parabola and straight line. While Tony recognised a relationship with graphs, tables and coordinates, he rejected links with equation, straight line and parabola. The presence of the parentheses probably accounts for his perceived link with "expansion". Finally, Patrick saw it as a "symbol" with links to "numerals" but nothing else.

The equation **y = 2x - 1** was associated with graphs, tables of values and straight lines by myself, Stephen and Tony, while Patrick linked it with numerals and "equals". The other equation (**2x - 1 = x + 7)**, however, led to more diverse responses. I saw it as an equation with elements of graph, symbol and table of values. Tony and Patrick linked it only to "equation" (or "equals" in the latter case), with Patrick recognising a numerical component. Both rejected association with graphical and tabular forms.

Stephen's response to the equation was somewhat surprising since, after rejecting association with function, "two pronumerals", parabola, straight line and the graphical representation, he explicitly linked it with the table of values. Further, this was a clear and deliberate relationship, which he identified several times in the task. In particular, he explicitly rejected the graphical form but chose the tabular. Until this point, the two representations had appeared to be interchangeable in Stephen's way of thinking (although the table had appeared a little more flexible). Clearly this item demonstrates that, to Stephen, an equation *is* a table of values and that this form, unlike the graphical form, may not require a specific format for the algebraic object. Stephen's diverse understanding of the tabular representation was further highlighted in his responses to that particular image, in which he associated it with function, equation, graph, parabola and two pronumerals. Responses for Tony and Patrick regarding the table of values card were more limited, relating it to coordinates and graphs but not equations in both cases.

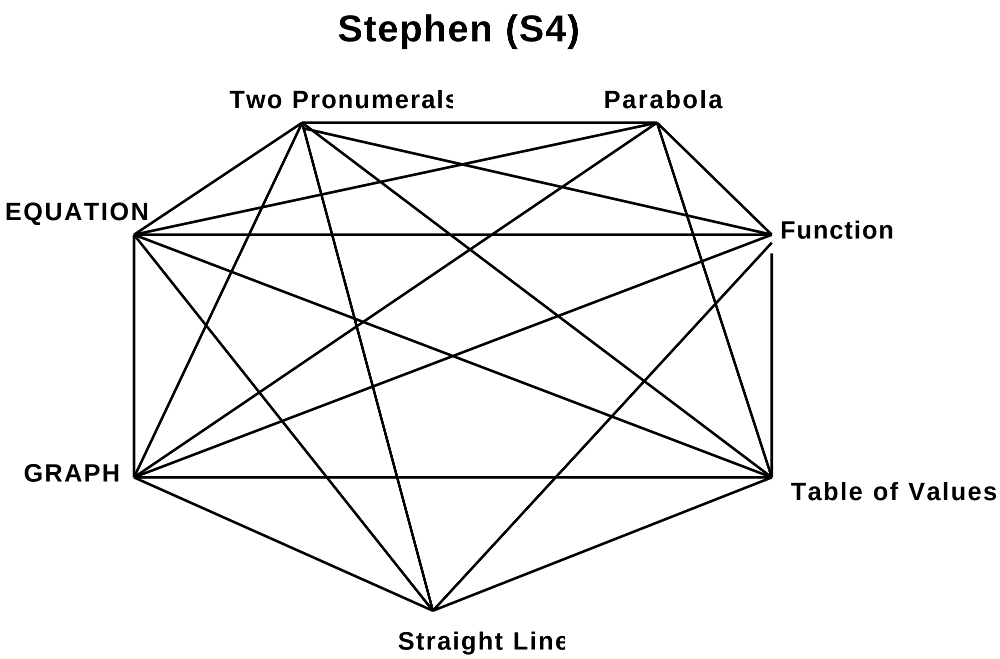
The networks of relationships derived from this task proved highly informative regarding the algebraic thinking of the various individuals involved. While the previous sorting tasks had allowed the identification of the various categories by which algebraic objects were conceptualised, this final task allowed these categories to be located within a dimensional space. Summary diagrams of these concept networks are presented in Figures 6.3 - 6.6.

Figure 6.3: Concept Network for SMA



The illustration of my own concept network depicts well-developed links for most descriptors. Some measure of this complexity is provided by the number of links associated with each node, or descriptor (note that the heavier lines indicate a strong connection across at least four different images; thinner lines indicate that the link was found in only one or two image cards). By this measure, the GRAPH and SYMBOL descriptors are the richest, associated with the widest range of algebraic concepts. The table of values is slightly less diverse, since the graph was found to be associated with a non-function, whereas the table of values representation was not. Similarly, EQUATIONS and EXPRESSIONS were strong but mutually exclusive categories, as were FUNCTIONS AND NON-FUNCTION. The network is suggestive of a relational understanding of algebra, in which the various components are perceived as meaningful, both in themselves and in relation to each other.

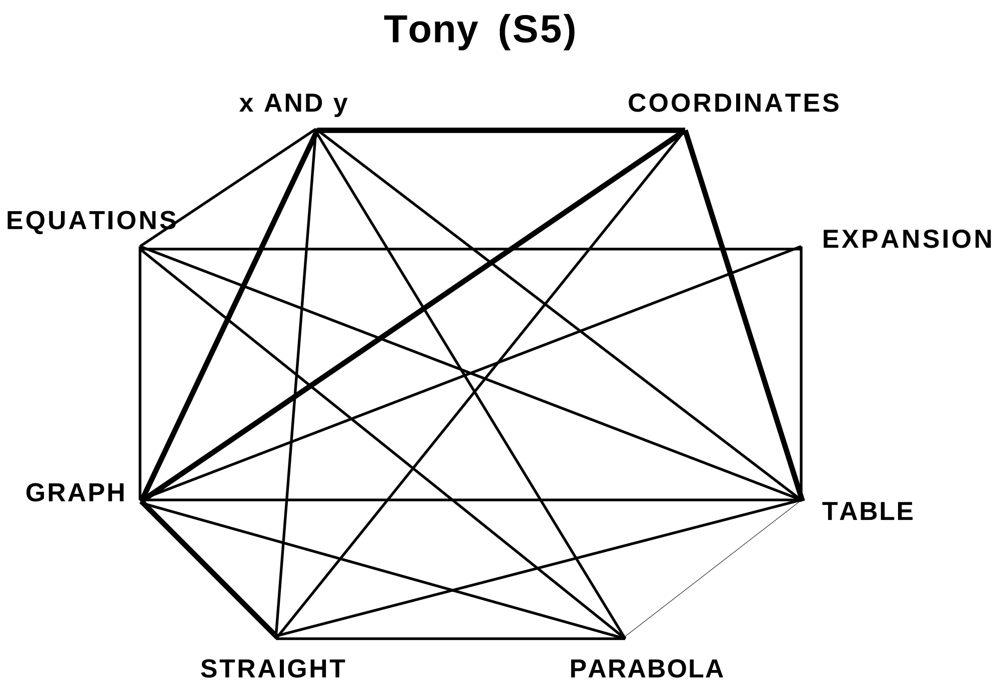
Figure 6.4: Concept Network for Stephen (S4)



Stephen's network of concepts displays well-developed links between all major categories, suggesting good algebraic understanding. His graphical thinking appears to be better developed than that associated with the table of values, with EQUATIONS and TWO PRONUMERALS being most extensively related. Stephen's thinking appears more "black and white" than that of the researcher - his concept links appear more of the "all or none" kind, suggesting that he distinguishes less clearly between them (as in his use of the terms "function" and "equation"). Although his understanding is best described as *relational*, it is clearly of a different order to that of myself.

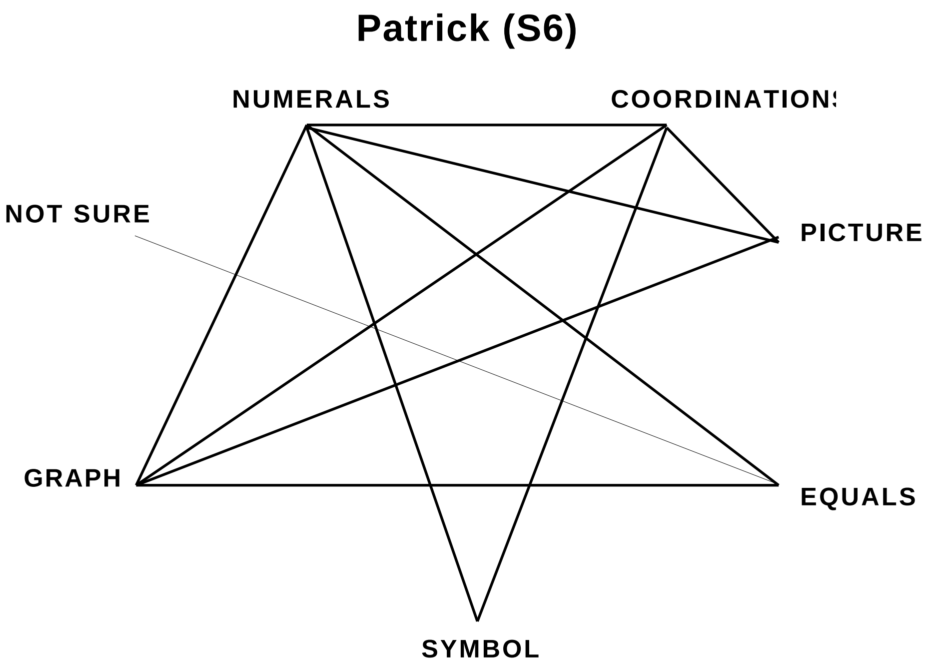
The differences between the concept network for Stephen and those for Tony and Patrick are immediately clear. While the younger students might have identified as many descriptors, these are poorly developed and associated constructs. Their relationships with other concepts is tenuous at best, illustrative of multistructural understanding at best.

Figure 6.5: Concept Network for Tony (S5)



It is hardly surprising to find Patrick's concept network to be even more limited than that of Tony, sure only that algebra involves pictures, graphs, numerals and symbols. Like Tony, repeated descriptors may be recognised: "x and y" and "coordinates" for Tony, "pictures" and "graphs" for Patrick. Patrick's few strong links are those between NUMERALS and SYMBOLS, and PICTURES and GRAPHS (demonstrating the symbol nature of algebraic forms). Clearly, for Tony, graphs are more meaningful objects than for Patrick, even to the recognition of the symbolic connections between equations and expressions and their graphical forms.

Figure 6.6: Concept Network for Patrick (S6)



The network diagrams shown above provide immediate visual clues as to the cognitive organisation of the participants. They illustrate both the nature and the relative strength of the relationships between the various constructs which go to make up each individual's cognitive network within the domain of algebra. Across all participants involved in the study, it is now possible to recognise important and detailed features of their algebraic thinking as a result of the intricate examination which has been described. The cognitive profiles which have been developed provide valuable insight into the algebraic thinking of these individuals which will give guidance in our further study of the role of computer tools in the algebra learning process.

An overview of algebraic thinking

It is useful at this point to draw together the extensive and varied findings related to algebraic thinking elicited so far. Although many aspects of such thinking are striking in their consistency across not only age and ability groupings, but even across student and preservice teacher groupings, other features define distinct individual conceptions of algebra for the various participants. It is possible at this stage to compare and contrast the algebraic thinking of the various groups of individuals who provided data for this study.

The preservice teachers displayed algebraic thinking which was generally versatile and reflected both traditional considerations - rules, formulas, variables - and some elements clearly drawn from their pedagogic studies, especially the role of patterns and concrete materials in algebra learning. Explicit recognition of formal aspects of algebra (especially the uniqueness property of functions, notions of domain and range and active conceptions of variable) suggest mature and well-connected understanding. At the same time, alternative representations appeared to play little part in their conceptions of algebra, with only A3 specifically mentioning graphs in relation to definitions of algebra, and tables of values in relation to functions. The grouping tasks indicated considerable variation in the depth of thinking inspired by the various images of algebra, from quite superficial connections on the part of A2 to relatively well-connected and rich groups displayed by both A5 and A6. Even at this level of mathematical achievement, there appears to exist quite significant diversity in the depth and connectedness of algebraic thinking among those who will soon be teaching the subject, with the majority clearly dominated by traditional perceptions and manipulative aspects of the study of algebra. As with the student participants, particular images were associated with strong and well-developed signal characteristics while others (especially the simple expression and table of values) appeared to be poorly connected in relation to both other representations and appropriate action strategies.

Of the student participants, Stephen and Andrea appeared most similar in the majority of their responses to both definition-based and image-based research activities. Each recognised algebra as serving both a representational role and an active "solving" role. Each showed good facility for relating symbolic and graphical information (correctly interpreting the algebraic expressions **4 - 3x** and **(x - 1)(x + 1)** in terms of their graphical features) and each explicitly recognised variables as representing multiple unknowns. In contrast, Ben saw algebra as fulfilling only a "solving" function, failing to recognise its usefulness in representing information in a variety of ways. Although Ben was functionally able to recognise and interpret the graphical representation, he appeared limited to this mode, unlike his peers who were able to operate across symbolic, graphical and numerical representations. Ben demonstrated a preference for graphical imagery when interpreting algebraic information, a reliance which proved useful for him on many occasions, but also hampered his ability to act and think flexibly when required.

Stephen displayed a preference for an active "function machine" image when thinking about and describing algebraic ideas. This image involved numbers "going in" to a symbolic expression and other numbers "coming out". Such thinking proved to be far more flexible than Ben's graphical image, readily supporting cross-representational thinking. In fact, the function-machine image appears to serve as a link across the three major representations, supporting versatile thinking which translates readily across numerical, graphical and symbolic forms. In Stephen's case, this powerful image was somewhat limited by his perceived need for a particular explicit notational form, particularly for functions (which required an "f(x) =" prefix) and graphs (requiring a "y =" prefix). Algebraic forms without such prefixes (such as **4 - 3x**) were emasculated for Stephen (and for others, including the preservice teachers), having nowhere to place the required "output" number, and so failing to support any sort of effective algebraic action. Interestingly, although Stephen indicated a willingness to see in such an expression an "implied equation" (solving it as if it read "4 - 3x = 0"), he was unable to concede that it could as easily possess an "implied y-value", allowing both graphical and tabular representations. This was *in spite of the fact* that both graph plotter and table of values utilities supported entry in this simpler expression form as well as the more usual "y =" form. It can only be assumed that Stephen's exposure to this flexible input feature was too limited to overcome his persistent belief that graphs and functions require particular algebraic formats. This belief may have been inadvertently reinforced by the instructional modules of the computer-based algebra learning environment, which automatically expressed algebraic objects to be graphed using the "y =" prefix, even when this was not present in the displayed form. Thus, clicking on an expression such as **4 - 3x** when encountered within the modules automatically led to the graphing of the equation **y = 4 - 3x**.

While Ben and Stephen displayed clear preferences for particular algebraic images, such thinking was not as clear for the other participants. Andrea demonstrated strong cross-representational facility, transferring meaning across all three representational modes, but she did not display the explicit "function-machine" imagery which distinguished Stephen's thinking. In fact, in the absence of an explicit preference for graphical thinking (as shown by Ben) or numerical thinking (as shown by Stephen), Andrea appeared comfortable with the symbolic form, from which she was readily able to deduce the other representations. Even the table of values (with its obvious link to the input-output form of the function-machine image for Stephen) was related by Andrea to the rule or equation which gave rise to it. Although logically one might expect that Stephen would display a preference for the table of values as a mathematical tool, based upon his preference for numerical imagery, he demonstrated limited use of this mode. In fact, its main function for Stephen appeared to lie in the display of the two "sides" of an equation, allowing solution by numerical methods. It is possible that his preference for an input-output metaphor for algebraic thinking may have left Stephen with "nowhere else to go" when confronted by a table of values, whereas for Andrea, such a form immediately suggested a symbolic rule or equation as its source.

Jane and the two younger student participants appeared to relate most strongly to the visual forms of algebra, particularly the symbolic notation by which it was most readily recognised. While Andrea saw beyond this symbolic form and related it to a relatively rich network of associations, this facility was all but absent among these younger respondents. For them, algebra was static - representational in nature only to the extent that it involved "letters standing for numbers". While graphs, ordered pairs and symbols were recognised as the components of algebra, the links between these elements were all but non-existent. Certain automated action processes were evident, but these were largely unsupported by understanding, and easily confused. The beginnings of links and relationships between the components of algebra was evident, but their development is clearly an extended process, even with the support of computer tools.

The study of algebraic thinking which has been described, then, suggests several important implications which may impact upon the use of available software tools. Individuals were shown to interpret different algebraic forms in a variety of ways. Certain forms displayed strong and consistent signal characters which readily led to action on the part of participants, both students and preservice teachers. Most notable of these were the two forms of equations encountered - **y = 2x -1** and **2x - 1 = x + 7**. The former was invariably associated with graphing, and the majority of participants demonstrated the ability to deduce useful graphical meanings (most particularly gradient and y-intercept information) from this algebraic form. The second induced in all participants an automatic action sequence, leading to the production of a “solution”. Both students and preservice teachers displayed some preference for solution by “physical” manipulation of terms, as opposed to the process of acting equally upon “both sides” of the equation.

While such forms displayed strong signal character, the simple algebraic expression **4 - 3x** induced frustration and confusion among even the most experienced participants. All expressed a desire to act in some way upon the expression, but were unable to do so with their existing repertoire of available mathematical actions. The expression **(x - 1)(x + 1)** was not associated with the same reactions, since it permitted a familiar mathematical action (expansion), and so induced a sense of closure. Since the majority of algebra software tools take such expressions as their principal objects of action, the lack of a strong and consistent signal character associated with this common form appears likely to significantly influence the use of computer tools.

Finally, preferred algebraic imagery was found to be of significance for several participants. While such images as the “function machine” were found to be flexible and to support cross-representational thinking far more effectively than a graphical image, it appears that over-reliance upon any one form may limit the effectiveness of algebraic thinking in general, and the use of representational software tools in particular.

In order to build a more complete picture of the use of available software tools by the participants in this study, however, it is necessary to go beyond considerations of algebraic thinking alone. The beliefs and attitudes of individuals regarding the learning of algebra must play a significant role in determining the nature and frequency of such use, and these must now be examined.

Seven
Thinking about Learning

If the primary units of analysis for the previous chapter were definitions and images, then this chapter is principally a study of beliefs. As Chapter Six sought to build up profiles of algebraic thinking by the various participants, so does this chapter examine aspects of pedagogical thinking by the students, preservice teachers and by the researcher. Beliefs and attitudes regarding the learning of mathematics must feature critically in understanding the responses of individuals to computer technology as a medium for algebra learning. This chapter examines beliefs and perceptions concerning:

- how algebra is best learned;
- features of a typical mathematics lesson;
- images and metaphors associated with more and less effective teaching practice, and
- constructivism in algebra learning.

From this data a second set of individual profiles may be developed which will further inform our consideration of algebra learning within a technology-rich environment.

How Do You Best Learn Algebra?

Throughout the entire data collection process, the question which became associated with the major theme of the study was the one which greeted participants each time they entered the computer-based instructional modules: *"How would you describe your beliefs about algebra and the ways you best learn it?"* The responses of the participants provide valuable insights into both their perceptions of effective learning and the nature of mathematics itself.

Stephen offered his responses to this key question several times over the period of the study:

1. I best learn algebra by repeating questions to understand why you do the question a particular way.
2. [Success in mathematics follows from] an understanding of what you want to find, how to go about finding the answer, [and] understanding why you use particular steps and formulas.
3. The best way to learn algebra is to see a diagram of the figure or graph and working out methods to find answers using rules and formulas.
4. The way I best learn algebra is in seeing graphs or curves of the equation and being showed how the rule is derived.
5. A successful maths lesson is when we were allowed to study and look over our work. The teacher would come around and help us with our problems and how to set out our answers. A student would help another and show how he approached the problem.
6. I learn it best by seeing the graph of an equation you have to work out and by knowing why you have to use a formula.

These comments spanned the two year period of Stephen's involvement with the study, from his commencement of Year 11 to the end of his Year 12 studies. Six critical elements may be discerned as characterising Stephen's beliefs about algebra learning:

1. Visualisation (especially using graphs),
2. "Being shown" how to arrive at a solution,
3. Repetition,
4. "Working out" a method for arriving at an answer,

5. "Knowing why" you use rules and formulas, and
6. Algebra is associated with memorising rules and formulas, and finding answers to problems.

The first two elements may be associated with a **passive** approach to learning, the second two with a more **active** involvement. All suggest common themes which may be observed to greater or lesser extents across the sample, from students to preservice teachers. In Stephen's case, the role of visualisation is clearly a dominant one, occurring as a repeated theme in several statements. The clear distinction between his first statement and the subsequent responses suggests a growth in metacognitive awareness, which probably arises as a direct result of his involvement in the research program. The regular requests for analysis of his own thinking about algebra and learning served to make him aware of his own learning and encouraged him to refine his response. The similarities of the latter responses suggest that his thinking on this matter had become relatively stable and that the long term involvement within a technologically rich learning environment may have done little to alter his thinking about learning other than to impress upon him the value of visualisation as a learning tool.

Stephen's reference to "knowing why" suggests that he perceives **understanding** as a central factor contributing to an effective learning experience. Within the context of his view of algebra as consisting of formula- and rule-based methods for finding answers to "problems", however, his "understanding" may be more instrumental than relational.

Finally, the perceived role of the teacher in effective learning is likely to be significant. Explicitly, successful learning was associated with a more student-centred learning context, in which the teacher assumed a supportive role. Throughout Stephen's responses, however, the teacher is never far from consideration - as the one who "shows how the rule is derived", who provides the reasons for the use of rules and formulas and, fundamentally, as the primary source of the "problems" by which algebra itself is defined. A potential conflict may be discerned between Stephen's association of successful learning with student-centred methods and yet his reliance upon the teacher as source of both that which is to be studied and the ways in which this study may best be achieved.

Stephen's peer, Ben, offered the following responses over a similarly protracted period of involvement within the research program:

1. I best learn algebra by revising the work. I've just got to chisel it into my memory because of the formulas.
2. [Success in working with functions and variables follows from] full understanding of what you are doing.
3. [Success in coordinate geometry follows from] good visualisation of what you are working out mathematically. Good understanding of the formulas.
4. I learn [algebra] best by continually just doing algebraic problems, until I get used to it, and start seeing different methods, different ways to go about it.
5. [A successful lesson was one in which] the teacher was able to get across the true mathematics and to help me to fully understand how to answer all the questions used throughout the lesson. By true mathematics I mean the substance behind it, why it's meant to do what it does.
6. [The three most important factors in learning algebra are] understanding, ability, memory for rules concerning algebra.

Once again, the common themes are those of repetition, understanding (but within a context of memorisation of rules), visualisation and teacher dominance. Once again, the teacher is perceived as the source of knowledge and skills which must be learned. There is an implication, too, in the last response that ability is a factor in determining success in algebra - that some people are naturally better than others.

Andrea offered two comments regarding algebra learning:

1. The best way to learn algebra is to memorise the equations and also know their purpose and use.
2. The best way to learn algebra is to learn the basic ideas first then take a few challenging questions according to the individual's level.

Present once again are the themes of understanding linked with memorisation and individual ability as a deciding factor in algebraic success. The second response, however, suggests some change in perception over the intervening period since the first. It appears to reflect elements of the van Hiele **stages of learning**, which begin with information and guided orientation, and progress later to free orientation which might be associated with Andrea's notion of challenge. Since these stages of learning provided the basis for the construction of the algebraic learning environment for this study, it does seem likely that she has adopted elements through her involvement with the program. Andrea's perception of challenge is notably absent from the views of the other respondents.

Like her conception of algebra, Jane's view of effective learning remained stable over the period of the study, centred upon repetition and memorisation.

1. ...doing lots of questions and examples that are given to me.
2. ...practise and revision.
3. ...revision, learning rules.
4. ...common sense, knowledge, and to use the skills you know.
5. ...you learn [algebra] best by revision, practise.
6. ...revision, practise and study.
7. I find it better when I work with the people around me. I like to have the time to think about what I am doing.
8. Revision, common sense and practice.

Jane's view of algebra learning appears to be largely passive, dependent upon the teacher, although she acknowledges that she works best in a social context.

For Tony, the element of innate ability is an immediate one, along with the now familiar themes of memorisation, understanding and repetition within a teacher-dependent learning context:

1. I think that some kids can achieve high scores in algebra right away and some can't.
2. [Success results from] knowing what the pronumerals and numbers stand for.
3. Learn all the formulas that go with [algebra].
4. Learning the rules involved and working it out.
5. [The three most important things for success in mathematics are] learning the rules and the formulas, studying hard and doing all your homework.
6. I best learn algebra by using the computer to show things graphically. This is better than when teachers blab on and write stuff on the board. The computer is most helpful in showing us alternate ways to do things and showing you graphically how to do the sums. The computer is least helpful when it crashes! When you get something wrong, the computer doesn't really help you - it can't work out what you did wrong. I like using a computer algebra program because it helps me to see what I have done wrong after I have done it.

Like Stephen, visualisation is a feature of Tony's learning of algebra; like Ben, Tony recognises the advantages of encountering **alternative** approaches to concepts and methods. The computer figures strongly in Tony's perception of effective learning, reflecting his experience with algebra software tools. He notes disadvantages as well as advantages in the use of technology in this context.

Patrick's experience of algebra learning had largely involved the use of computer-based concrete manipulatives, and some limited use of computer algebra software. Like Tony, he sees innate ability as a significant factor in deciding success or failure in algebra learning, but he reacts positively to the concrete (visual) approach:

1. I think it is important to be able to understand algebra to do it.
2. ...it is best to learn it at your own pace because taking it too fast could ruin your image of algebra.
3. [Success in algebra follows from] trying different kinds and using the knowledge of those who offer assistance to you.
4. The best way to learn it is not imagery (in your head) but in visual maths, with objects in front of you.
5. To be successful in algebra you must KNOW what you're doing and BELIEVE that you can actually do it. When I used the computer to solve some equations I didn't learn anything because the computer did it for me leaving me only with an answer not the knowledge of how to do it.
6. Algebra is using maths in a different way and it is best learned by using solid and visual objects.
7. You need to be taught well the first time around or you won't understand as well when older. You should have visual objects and you should be good at numbers.

Patrick offers some unique perspectives, in addition to echoing the themes already observed. He notes, for example, that the **pace** of instruction is a significant factor, and that **confidence** in your own abilities influences learning. He observes that skill in numeracy is an important precursor to algebraic facility.

The view of algebra which emerges from the responses of the six students in the study is one which is surprisingly consistent. Successful algebra learning, for all students, is associated with **understanding**, but this appears to be in the sense of "knowing what to do" rather than "knowing why it is done". This form of understanding for the students is perceived as resulting from repetition, leading to memorisation of rules and formulas. While it may result from the efforts of the student ("working out") it appears more likely to involve teacher intervention, in which the student plays a largely passive role ("being shown"). Associated with the passive role of "being shown" appears to be strong general reliance upon visualisation and frequent reference to the graphical representation of algebraic ideas. Although the teacher is generally recognised as the source of both knowledge and method in algebra learning, students also commonly make reference to a preference for working within a social context, learning through interaction with their peers.

The preservice teachers appeared to interpret understanding differently from the students. While the latter group saw it as following from the learning of rules and formulas, the preservice teachers generally saw the two as incompatible. For A1, for example:

> Algebra and the way it is taught is [sic] currently being debated by educators - are we teaching algebra effectively or are we taking the easy approach to teaching algebra by teaching rules and not understanding? I believe understanding is crucial but in schools we are told that we don't have time to explore algebra and we must teach the rules and move on ... Although it might be easier to just teach the rules in the earlier years it would be more effective and more beneficial in the long term to encourage and facilitate the students to develop an understanding of variable and the relationships between variables in algebra.

A3 points out that "without an understanding of the concept of a variable, the study of algebra is meaningless and simply becomes the learning of rules and the manipulation of meaningless expressions". A6 echoes these sentiments:

> Overall, I think that algebra is best learnt when it is not seen as a set of rules. I think that teachers need to concentrate on the real mathematics involved rather than blind manipulations. The basis of this is giving proper meaning to variables and a focus on why different things are able to be done.

The preservice teachers had clearly been exposed to alternative methods for the teaching of algebra, while the students had only their school experience to reflect upon. This experience appears to dominate their perceptions of algebra learning - although the technology-enriched program of this study attempted to engage the participants in learning which was meaningful and context-bound, they remain locked in a view of algebra as a collection of rules and formulas, where success is a direct result of memorisation and rote learning.

Aside from their different interpretations of "understanding", the two groups (students and preservice teachers) appear to share many common perceptions regarding effective algebra learning. A2 points out that algebra learning "is best done by having as many practical examples and activities as possible, followed by theory and questions. It must be done in a logical and sequential way or pupils will be lost and not understand what is going on". A5 states clearly, "I think students best learn by gradually doing each section. Some students need to learn the rules by rote, others learn by using different methods". She goes on to point out the importance of repetition, sequencing and "being shown":

> I still think a lot of students learn algebra by learning the rule by rote. They don't fully understand the reasons for what they do. Personally I think the students would learn better by being given a thorough explanation and many worked examples. The students then need to do a lot of practice of what they have learnt. They should learn one section at a time, so they don't get confused. Then as they progress they will be able to add all the skills and rules they have learnt to solve more complicated problems.

Although the preservice teachers recommend concrete methods and number patterns for developing algebraic understanding, they appear in general to reflect traditional values associated with algebra learning - that success follows from memorisation and repeated exercises. Their view of learning appears to be strongly teacher-centred, reflecting the view of the students. Only one of the preservice teachers (A3) mentions group work, and that is in a limited way ("The ability to work in group or paired exercises would be advantageous in the early stages of algebra"). Their understanding of the algebra learning process suggests a hierarchy of skills, built upon numerical foundations (echoing Patrick's comment) and requiring the ability to interpret symbols.

A1 defines a "successful lesson" as follows:

> A lesson is successful when the students understand the content of the lesson and are able to do the exercises without great difficulty. However, it is hard to pinpoint what made the lesson successful. I think it basically comes down to a number of uncontrollable factors:
> - the way the teacher explains the examples and the use of the formula/rule, sometimes the students understand straightaway what the teacher is talking about and other times what the teacher says goes right over the students' heads.
> - time of day - morning lessons are always more successful than afternoon lessons and Monday's lessons are more successful than Friday's lessons.
> - the content area - is it interesting to the students, is it relevant to them?

Even within an increasingly student-centred context, the successful algebra lesson revolves around the activities of the teacher (A2):

> The things that make a lesson successful are good easy to follow explanations, followed up by a student explanation to see if they are following everything that is going on. The examples used in demonstrating the examples [sic] are relevant to the pupils' level and reinforce the explanation. The examples would show step-by-step the way the concept works and would be easy to follow. The students could do these up on the board, as the students often enjoy this. A quick quiz would show to the teacher if the pupils understand what it is they are meant to be doing. A discussion on the topic would show what misconceptions the pupils have and a game on the topic makes the pupils more interested as they think they are not "doing maths".

The differences between the two groups appear minimal in comparison with their similarities. Both groups make specific mention of teacher-centred methods which involve demonstration, repetition and carefully sequenced exercises leading to memorisation of skills of manipulation, generally associated with success within this domain. The common views of the nature of algebra and the ways in which it may best be learned span both groups of participants, and appear to suggest the existence of what might be termed a **mathematics learning culture** (where the term "culture" is used in the sense of shared meaning and experience). The existence of such a phenomenon has significant implications for the use of computer technology if it is as pervasive and uniform as it appears. Further evidence as to the existence and nature of such a culture may be gained by triangulating the various sources of data related to algebra learning which this study makes available, particularly responses to the Constructivist Learning Environment Scale and the “grand tour” question, “Describe a typical mathematics lesson”.

A Typical Mathematics Lesson

When asked to describe “a typical mathematics lesson”, the responses of the various student participants were almost identical - after a momentary pause, each began to write, confident at last that they were being asked about something in which *they* were the experts, about which they could respond confidently from experience. The various responses are quoted below, providing as they do valuable insights into the learning experience which all appeared to share and which, more than any other group of data in this study, offers convincing evidence for the existence of a pervasive culture of mathematics learning.

My own response was among the most extensive and suggestive of a relational level of thinking, as opposed to the largely multistructural responses of the younger participants.

> A typical maths lesson begins with work set by the teacher - perhaps a few quick review questions, or review of the previous night’s homework. This ‘housekeeping’ is an important part of beginning the lesson, preparing students for work.
>
> The body of the lesson typically involves the teacher demonstrating some new concept or technique, and then (after questions from the students) having them practise this - usually using questions from a textbook. The teacher will usually move around the class while this deskwork is in progress, assisting, answering questions and also keeping students on task.
>
> Occasionally, the teacher will call attention back to the front to explain further some point that may need clarifying, or to set some new work.

> In the last few minutes of the lesson, opportunities are again provided for questions, homework is set, and students continue with the work or begin the homework.

For Stephen, the components of the typical lesson are distinct and unrelated parts which occur within a specified sequence:

> A typical maths lesson begins with any problems from homework to be answered and explained. Then the teacher will show us the new work and explain it. Then we will be shown a few examples. We will go on with the new work and if we need any help the teacher will show us what to do.

This same sequential element is present in Jane's description, although she makes more mention of physical spatial components:

> The teacher begins by writing questions on the board - sometimes general questions, sometimes related to what we are doing. We answer the questions, correct them, then we might work on questions out of the textbook or we might answer questions from the homework. If work is set then the teacher moves around the class; if not, then she stays out the front. At the end we are set homework for the night.

Ben responded twice to this item, demonstrating the sequential multistructural format already seen, and later a more relational level of thinking about this subject, offering justification for the various components of his typical lesson.

> A student is taught set mathematics from a teacher. It begins with a couple of warmup questions followed by work we have been doing on a set topic, half of which is taught and the other half comes from a text book. Then the rest of the lesson is spent working from our textbook. Then we are given some homework.
>
> It starts with a warmup question based on the topic we are currently studying, then we continue on our selected topic learning what we need to know, then we are given homework to help us develop our new skills. The teacher explains the reasons behind the mathematics involved and helps us to learn the techniques involved. The work comes out of a textbook.

Although generally more detailed than the student responses, the replies of the preservice teachers to this question might have been drawn from the same classroom. A5 offers a succinct account which effectively captures the recurrent elements of the previous responses:

> A typical Maths lesson is one in which a new concept is taught at the beginning of the lesson, then it is practised by the students throughout the lesson. The practice consists of examples from the textbook or a worksheet.

A6 offers a more detailed description, but one easily recognisable as containing the same themes:

> From my experience, a typical maths lesson begins with the teacher explaining new work or revising previous work. The students generally have the opportunity to ask questions during this process. After the explanations the students are usually asked to complete a set of exercises. These exercises typically come from a text book and involve drill and practice style learning. The students will usually work individually although some communication is often allowed. The teaching is almost always 'chalk and talk' with the emphasis being on instruction rather than discovery.

Eleven major descriptors may be identified from the responses to this aspect of the study, and these are summarised and displayed with their respective respondents in Table 7.1.

Table 7.1

Descriptors for a Typical Mathematics Lesson

	A1	A2	A3	A4	A5	A6	Ben	Jane	Stephen	SMA
Teacher Explanation	√	√	√	√	√	√	√		√	√
Tchr. Demonstration	√	√	√	√	√	√	√		√	√
Review / Examples	√	√	√	√		√	√	√	√	√
Homework	√			√			√	√	√	√
Textbook					√	√	√	√		√
Teacher Questions	√	√	√	√		√	√	√		√
Student Questions				√		√				√
Student Deskwork	√	√	√	√	√	√	√	√	√	√
Teacher at front			√	√				√		√
Teacher moving								√		√
Discipline ("on task")								√		√

There can be no more convincing argument for the existence of a mathematics learning culture than that presented in these vignettes. Although some are more detailed than others, all could have been descriptions of the same mathematics classroom, taught by the same mathematics teacher. The power of such a culture to influence the participants on both sides of the desk - teachers as well as students - must be recognised if the impact of any innovation is to be understood. This is a culture which is dominated by the influence of the teacher - each and every description begins with a reference - either direct or implied - to the actions of the teacher. The active role of the teacher is strongly contrasted with the passive role of the students, who are required to work through set exercises and examples - most commonly from a text book - in order to learn how to answer the “problems” and “questions” which appear to define the subject area.

A view of mathematics as answer-oriented is significant within the context of computer tools which appear to be of most use within an open-ended exploratory role - a role which appears to have little or no value within a culture dominated by the finding of a predefined “answer” through a specified sequence of steps involving some manipulation of rules and formulas. This pedagogic role of technology is explored in greater detail in the following chapter; the nature of the mathematics learning culture is examined more closely now through consideration of images and metaphors associated with more and less effective teaching.

Pedagogical Images and Attitudes

This research activity involved participants in the verbal description of a series of "teacher roles", followed by their categorisation into those associated with more and less effective teaching. Associated as it was specifically with teaching, not all participants were required to complete this task; all Group A preservice teachers provided responses, as did the students Ben and Tony, and myself, as researcher. Table 7.2 summarises those roles considered to be associated with successful teaching (√) and those associated with unsuccessful teaching (**x**) for each respondent.

Table 7.2

Successful and Unsuccessful Teacher Roles

Teacher as...	SMA	Ben	Tony	A1	A2	A3	A4	A5	A6
Entertainer	√	√	√	√	√	√	√	√	
Police Officer	**x**	**x**	**x**	√	√ **x**	√ **x**	√	√	**x**
Gardener	√	√		√	√	√	√	√	
Captain of the Ship	√			√		√ **x**	√	√	√
Travel Agent	√	√	√	√	√		√	**x**	√
Social Secretary	**x**		**x**	**x**	**x**		**x**	**x**	
Tour Guide	√	√	√ **x**	√	√	√	√	**x**	**x**
Administrator	√	**x**	√	**x**	√	√ **x**	√	**x**	√
'The Boss'	√		√ **x**	**x**	**x**	√ **x**	**x**	√	**x**
Other?		***student***	***Diving instruct***		***Couns-ellor***		***Politic-ian***		

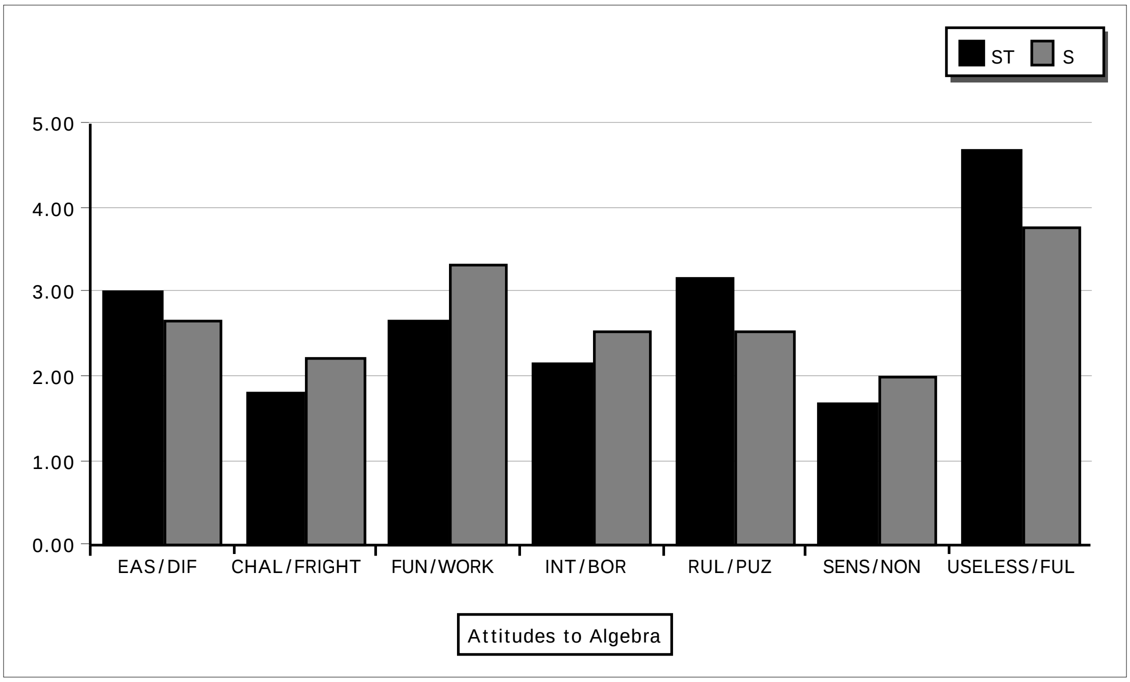
Responses to this task were varied, some participants recognising that certain roles may have both positive and negative associations with regard to teaching. Generally, however, those roles most associated with discipline and authority (Police Officer, Captain of the Ship and 'The Boss') were perceived as less successful than those which involved making the learning interesting (Entertainer and Travel Agent) and guiding rather than forcing the learners (Gardener).

Participants' own categories also proved revealing of their perceptions of teaching and learning. Ben's "student" role ("no-one knows everything and they are still learning") appeared as a negative response to teacher authority, while Tony's "diving instructor" appears relevant in a context of mathematics as a specific set of skills to be learned. For A2, "the counsellor is the person who the pupils can turn to in times of crisis and is one who can offer solutions to their problems. The counsellor shows how to deal with situations effectively". Such a caring view of teaching appears not to be adequately covered by the other roles, and implies a duty of care associated with teaching beyond that of simply imparting knowledge and skills. The "politician" role described by A4 related to the public nature of teaching - "A teacher like a politician must remember that they are always in the public eye and must act accordingly: a teacher is a role model in and out of the class". As with the counsellor, this view of teaching extends beyond pedagogical responsibilities.

Once more we find that the various participants appear to share more common views than differences. The teacher is recognised as the critical factor in mathematics learning and, although some resentment is evident related to abuse of authority ("a teacher... should not be a gestapo-type disciplinarian" (A4) and "teacher abusing authority given to them" (Ben)), the effective teacher is nonetheless one who is "in charge" and responsible for student learning.

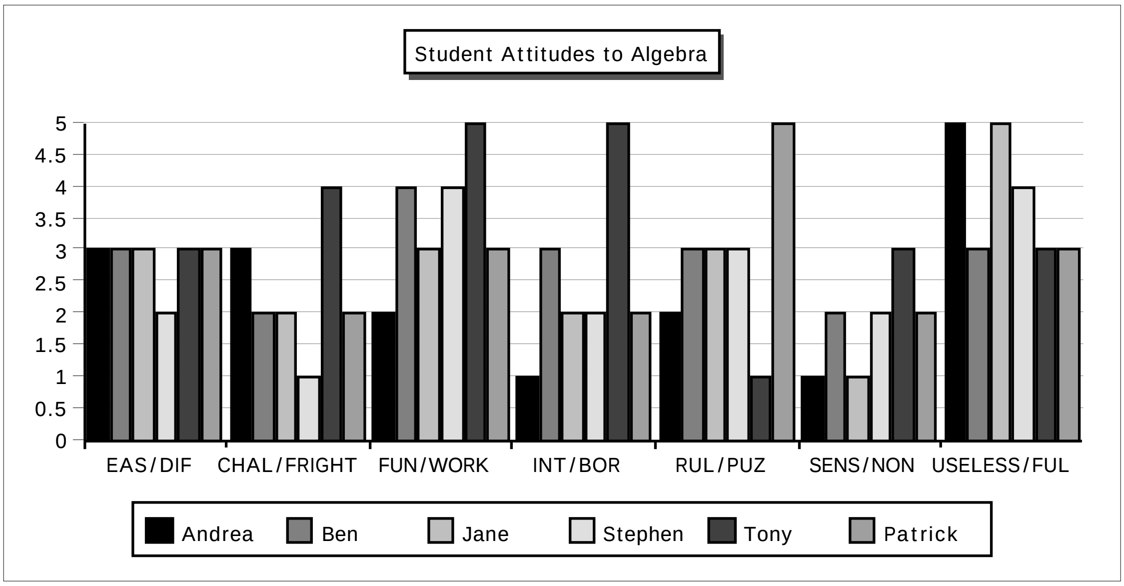
The use of images and metaphors associated with teaching appeared to induce in some respondents emotive responses which touched upon attitudes towards both teaching and learning, and mathematics itself. These were particularly evident in association with issues of discipline and control. Attitudes to algebra learning were recorded using a simple seven item Likert-scale task (Appendix C), and the responses from the two groups were averaged and graphed in Figure 7.1.

Figure 7.1: Attitudes to Algebra: Comparison of averaged responses from student teachers and students.



Almost all responses indicated positive attitudes towards algebra. Visually, bars below a score of 3.00 may be considered positive. The major exception to this tendency occurred on the last item - both groups considered algebra to be more **useful** than **useless**, with the preservice teachers being more positive in this regard than the students. This supports the earlier data in which applications of algebra were virtually unknown and its primary purpose is seen as supporting further study of mathematics. Students saw algebra as more **work** than **fun**, while student teachers perceived of it as more like **rules** than **puzzles**. Overall, the students appeared to display more positive attitudes to algebra than the preservice teachers, but with few overall differences between the two groups.

Figure 7.2: Student Attitudes to Algebra



Among the students themselves, there is again general agreement regarding perceptions of algebra as more positive than negative - more “challenging” than “frightening”, more “sensible” than “nonsense”. Most negative of the older students is Ben (S2) who finds algebra more **boring**, less sensible and less **useful** than any of his older peers. Most positive is Andrea, who appears to both enjoy algebra and find it interesting, but at the same time recognises it as both challenging and frightening. At the time of completing this task, Tony's attitudes towards mathematics were quite negative, although there was evidence of a more positive swing over the period of the study. Patrick, not surprisingly, saw algebra strongly in terms of **puzzles** and more likely to be **nonsense** than **sensible.**

These attitudinal responses help to place the various perceptions of both mathematical and pedagogical aspects of algebra within context for the different participants. They further reinforce the detailed profiles being constructed regarding elements of thinking and action which may influence their use of available computer tools.

Constructivism in Algebra Learning

The most detailed profile data available concerning pedagogical thinking by the respondents was that derived from the version of the Constructivist Learning Environment Scale (CLES) adapted for use in this study. Participants were asked to choose those responses which they would most *prefer* when learning algebra. The twenty-eight items of the scale represented four major groupings characterised by the authors as representing significant aspects of constructivist thinking and practise - **autonomy, negotiation, prior knowledge** and **student-centredness** (Appendix B provides a complete listing of these items). Some aspects of these factors have already been identified with regard to the student and preservice teacher participants. The individual responses to the items of this scale offer a further valuable triangulation method by which both validity and reliability of the associated data may be better judged.

Table 7.3

Responses of Group A student teachers, students and researcher to the Constructivist Learning Environment Scale

	A1	A2	A3	A4	A5	A6	Andrea	Ben	Jane	Stephen	Tony	Patrick	SMA
N1	+	+	+		+			+			+		++
(N5)	+	+	+		+		—	++	—	+	++	+	+
(N9)	++	+	++	+		+	+	++	+	+	++	+	+
N13	+	+	++	+		+	+	++	+	+	+	+	+
N17		+	+	+	—	+		+			+	+	+
N21		+	+	+	—	+		+			+	+	+
(N25)	+	+	+	++		+	+	+	++	+	+	—	++
PK2		+	++		++		+			+	++	+	+
PK6	+	+	+	—	+		++	+		+	+	+	+
(PK10)	++	++	++	++	++	++	++	++	++	++	++	++	++
PK14	+		+	++	+	+	+	—		+	+		++
PK18	+	+		++	++	++	+	+	+	+	++	++	+
(PK22)	++	+	— —	+	++	++		—	+		++		+
(PK26)	++	+	++	++	++	++		+	++	+	++	++	+
A3		+	+	+		—			+	++	+	++	++
A7	—			++			+		—	+	—	++	+
A11				+				—	—			++	+
A15	—		+	+	+	+		+	+		+	+	
A19				++	—		+		+	+	—	+	+
A23				+	—			+		+		+	+
A27	—			+			—			—	+	+	+
(SC4)		—	—	— —			—		+	—	—		
(SC8)	+	—			+	—	—	+	+				
(SC12)	—		— —	—	+	— —				+	—		
(SC16)	+	—	—			—	— —		—		—		—
(SC20)			—	—		— —	— —	—	—	—	— —		
(SC24)		—	—			+	— —	+		—	—	—	—
(SC28)				—	—	— —	— —	—	—	— —	— —		

KEY: Item numbers are prefixed with the scale type (N: Negotiation, PK: Prior Knowledge, A: Autonomy and SC: Student centred). ++ indicates a strong positive response, + a positive response, — denotes a negative response, — — a strong negative response and neutral responses are blank.
NEGATIVE items from the scale are parenthesised; responses to these items were reversed for purposes of analysis and presentation.

Table 7.3 provides a visual display of participant responses to the CLES items which reveals an interesting pattern of distribution. All items associated with the categories of **negotiation** and **prior knowledge** drew almost entirely positive responses; items associated with **autonomy** were largely neutral, and the **student-centred** items drew largely negative responses from all participants. This is supportive of the findings already discussed relating to pedagogical thinking by the respondents: algebra learning is perceived to be (and in this case, *preferred* to be) a **teacher-dominated** activity, in which student **autonomy** is not valued highly as a factor associated with success; in general, while most participants prefer to learn socially (the majority of **negotiation** items reflect this factor), **prior knowledge** is also considered an important feature, associated with rote learning and memorisation of rules and formulas already noted. Once more, strong evidence exists for a high level of uniformity across all respondents, suggestive of the common culture of mathematics learning which has been proposed.

Those items which drew the most uniform responses are italicised in the table. One item (PK10) drew the same strongly positive response from all participants: When *I do algebra, I would prefer there to be NOT enough time to really think*. The negative items (such as this one) had their responses reversed in order that high scores be associated with positive responses and low scores with negative responses. In this case, every respondent chose the same option - never.

Other items which drew almost entirely positive responses were:

N9: *When I do algebra, I would prefer NOT to be aware of other students' ideas* (Negative item - score reversed).

N13: *When I do algebra, I would prefer to talk with other students about the most sensible way of solving a problem.*
PK26: *When I do algebra, I would prefer that the things I learn about are NOT really interesting.*

Each of these strong responses related to a preference by the majority of participants for social interaction as a part of the learning process and a natural desire to be interested and not to be rushed and confused in learning.

Items which drew the most strongly non-constructivist responses from most participants were all classified as measuring "student-centredness":
SC4: *When I do algebra, I would prefer that the teacher gives me problems to investigate.*
SC16: *When I do algebra, I would prefer that the teacher expects me to remember things I learned in past lessons.*
SC20: *When I do algebra, I would prefer to learn the teacher's method for doing investigations.*
SC28: *When I do algebra, I would prefer that the teacher shows me the correct method for solving problems.*

These items paint a clear picture which reinforces the perception of algebra learning already encountered throughout the data. Both students and preservice teachers indicate a preference for an algebra learning environment in which the teacher is the source of both content and method, in which there is a "correct method for solving problems" and that this is the teacher's method. This is a learning environment in which the responsibility for learning rests squarely upon the shoulders of the teacher, and in which students are passive participants, waiting to be told and shown what to do.

Differences between the various individuals are not great. Of the students, Ben, Tony and Patrick emerge as rating **negotiation** very highly, with Andrea, Jane and, to a lesser extent, Stephen suggesting a tendency to rely less upon their peers. At the same time, Andrea and Tony indicate the strongest tendency towards teacher-dependence from their responses to the **student-centred** items. Only three respondents (A4, Patrick and myself) indicated a strong tendency towards **autonomy**; most others gave neutral responses to these items.

On item A19 (*When I do algebra, I would prefer that I decide if my solutions make sense*), most students responded in a mildly positive way, while the preservice teachers gave a largely negative response. Their greater mathematical experience had, perhaps, reinforced for them that their judgements are not to be trusted, allowing little room for confidence in their answers. Once again, the students' responsibility for learning is abdicated in favour of external sources of authority - presumably teacher and textbook.

The picture we have painted of the current state of algebra learning is not an optimistic one in terms of constructivist principles. There is clear and diverse evidence for the existence of a culture of mathematics learning within this domain which appears to exist across all participants in this study. It is a culture which values rote learning and memorisation of rules and formulas, where understanding is largely instrumental and in which students prefer to be passive recipients of knowledge and procedures transmitted through teacher and textbook.

There appears to be little motivation for independent study or exploration and students appear unlikely to value or even to trust their own answers to the external “problems” and “questions” by which this domain is defined.

An Overview of Pedagogical Thinking

The learning of algebra, for both students and preservice teachers in this study, is a teacher-dominated activity. It is most often associated with memorisation of rules and formulas through repetition and, subsequently, evaluated by reference to instrumental rather than relational measures of understanding. Mathematics (and algebra in particular) is associated with a fixed body of knowledge and an associated collection of actions and sequential processes which are used to arrive at a predetermined solution through a well-defined series of steps. While algebra is strongly associated with finding answers to problems, these problems are unlikely to relate to non-mathematical areas or real-life concerns; they are largely provided by teacher and textbook. Instruction in algebra is linear and sequential, as students seek to replicate and automate the skills and procedures by which the subject is most readily recognised.

Within this rigid framework, individual differences were observed in relation to a number of variables. Perceived responsibility for learning was associated with active and passive conceptions of the learning of algebra, with students such as Stephen and Andrea displaying a preference for “working out” solutions, while Ben and Jane preferred to “be shown”. Across all participants, however, the onus of responsibility was seen to lie with the teacher, who is expected to “show” the students "how to do it" and to manage critical learning variables such as sequencing and pace. While teachers might be expected to try to make the learning experience relevant and interesting, their primary responsibility was to lead and enforce learning, even though this was sometimes associated with abuse of authority. Attitudes to algebra varied widely for the student participants, with Ben and Tony associated with the most negative attitudes, and Andrea displaying the most consistently positive responses. Stephen was the only student who rated algebra as more easy than difficult; all other responses were neutral in this regard, suggesting limited confidence in their own abilities.

With regard to constructivist practices and beliefs, the preservice teachers were more strongly positive than the students in relation to both negotiation and prior knowledge scales. Ben and Tony again stand out, rating these factors more highly than their student peers. Least student-centred of the students were Andrea and Tony, although overall responses suggested a preference for teacher-dominance. Patrick's response to the autonomy items (positive) and the student-centred items (neutral) distinguished him from his peers: it seems probable that his limited experience of formal algebra learning situations had failed to initiate him into the dominant culture: he expected that *he* should be responsible for his own learning, rather than the teacher. In other respects, responses were strikingly uniform, and such a powerful culture must be a clear influence upon perceptions of the role of technology within algebra learning situations.

Eight
Using the Tools

The deliberate and goal-directed use of tools has always been an essential feature of human activity and a means by which we may most readily be distinguished from animal species. As means of enhancing, not only physical, but cognitive activity, tool use becomes a defining feature of that which is most uniquely human. Vygotsky begins his book, *Thought and Language* (1962) with a quote (in Latin) from Francis Bacon, broadly translated by Bruner (1986) as:

> Neither hand nor mind alone, left to itself, would amount to much. And what are these additional [tools] that perfect them? (p. 72)

Vygotsky is referring here to **thought** and **language** and the cognitive aids used to support these. Most importantly, language may be perceived as a cognitive tool which aids thinking (Vygotsky, 1978):

> Children solve practical tasks with the help of their speech, as well as their eyes and hands. (p. 86)

As described by Bruner, “language is... a way of sorting out one’s thoughts about things” (Bruner, 1986, p. 72). To Vygotsky, language is an example of a **sign system**. Sign systems include spoken and written language, and the number and symbol systems of mathematics - all used as tools to aid thinking. In fact, for Vygotsky, it was this use of sign systems which distinguished humans from animals (Vygotsky, 1978):

> Comparative analysis shows that such activity is absent from even the highest species of animals; we believe that these sign operations are the product of specific conditions of **social** development. (p. 39)

The use of tools serves to extend and enhance human capability, making possible that which otherwise may have been too difficult. At the same time, effective tool use requires a certain level of skill, both with regard to the use of the tool itself, and the object to be acted upon. In the case of the mathematical software tools considered here, effective use requires not only proficiency with regard to the computer application, but also prerequisite mathematical knowledge and skill. Having access to a computer algebra package, for example, will not permit students to act mathematically and meaningfully far beyond their current capabilities. Nor will they be in a position to choose a tool if they are unfamiliar with that tool or with the mathematical process in question.

Vygotsky's **Zone of Proximal Development** is particularly relevant in this context. While external factors (such as computer tools) may serve to extend and enhance individual functioning, the limits to which this will occur may be expected to be well-defined by the current state of the individual and by the nature and limitations of the tool. Computer tools for algebra serve two fundamental purposes - they permit **representation** and **manipulation** of algebraic ideas. While their representational capabilities are readily recognised as powerful and unique, the latter function appears to cross over boundaries set by centuries of mathematical tradition.

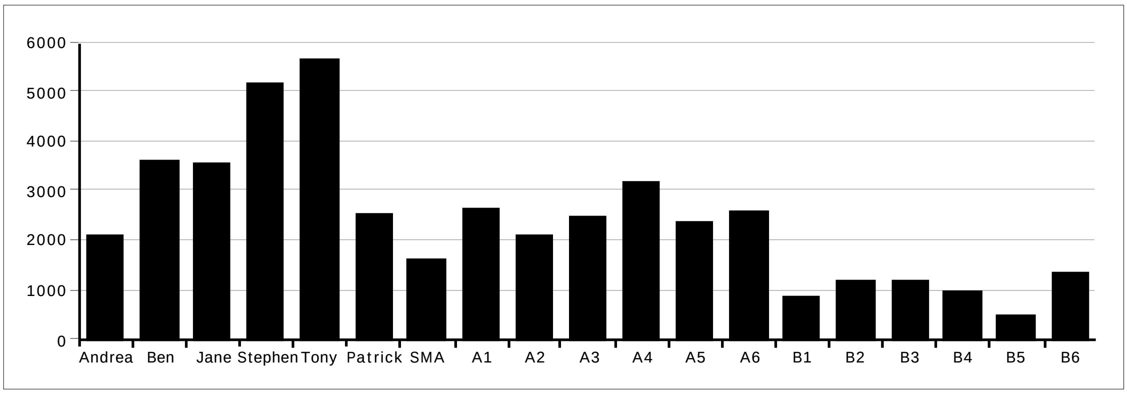
The tools used to enhance and enable mathematical activity have long been dominated by the sign and symbol systems by which this discipline is most readily recognised, aided externally by little more than pen, paper and, in certain circumstances, restricted access to geometrical tools. Mathematics as a purely cognitive activity has always been highly prized and nowhere more so than in schools. The advent of calculator and computer into mathematics learning has been met with a cautious and grudging acceptance, and the use of technology permitted only within very restricted boundaries.

This chapter examines the ways in which the participants engaged in the use of the various computer tools available to them within the confines of the study. It describes the context of this use, and the subsequent responses and reflections of the various participants. It seeks to make explicit those features of mathematical tool use which were most significant to the students and preservice teachers, and consequently to identify factors which may have served as both impediments and enhancements in their learning of algebra within a technology-rich environment. When considered within the context of the analyses of algebraic and pedagogical thinking described previously, this chapter sets the scene for the development of the grounded theory of mathematical software use which follows.

Frequency of Tool Use

The individuals who participated in this study did so over varying periods of time, and engaged in a range of different activities. Consequently, a measure of the extent of their interaction with the available software tools must take this variance into account. A simple gauge as to the extent of the contribution of each individual to the data record may be derived from the number of **text units** for each, as determined by the qualitative analysis program, *NUD•IST*. Each text unit is essentially a single line of text, with up to eighty characters per line. *NUD•IST* automatically processes each document into such units, and numbers each for ease of analysis. A visual display of this measure is provided in Figure 8.1 (the numerical data for each of the graphs in this chapter is available in Appendix F). It is clear from this display that Stephen and Tony provided the most detailed and extensive information, and the Group B preservice teachers the shortest term of interaction. Much of the "bulk" of the research record may be attributed to the various research questions and tasks designed to provide the background information already examined. Actual tool use must be considered within this context.

FIGURE 8.1: Number of Text Units for Participants



The frequency of tool use in this study is measured in terms of what are labelled here as **incidents**. An incident of tool use involves the deliberate selection of a particular software application for a mathematical or pedagogical purpose. Incidents do not include occasions when a tool is selected but not acted upon, as occurred when some respondents were familiarising themselves with the program. The number of incidents ranged from 4 to 63, and are displayed in Figure 8.2. Relative to the number of text units for each respondent, however, the incidence of tool use across the participants is more clearly conveyed in Figure 8.3. While the students provide the most frequent examples of tools use in terms of raw numbers of incidents (as shown in Figure 8.2), the preservice teachers in Group B used the tools far more frequently when their more limited involvement is compensated for. Of the students, Ben, Andrea and Stephen were the most frequent users of available tools.

Figure 8.2: Frequency of Tool Use across Participants (Number of incidents)

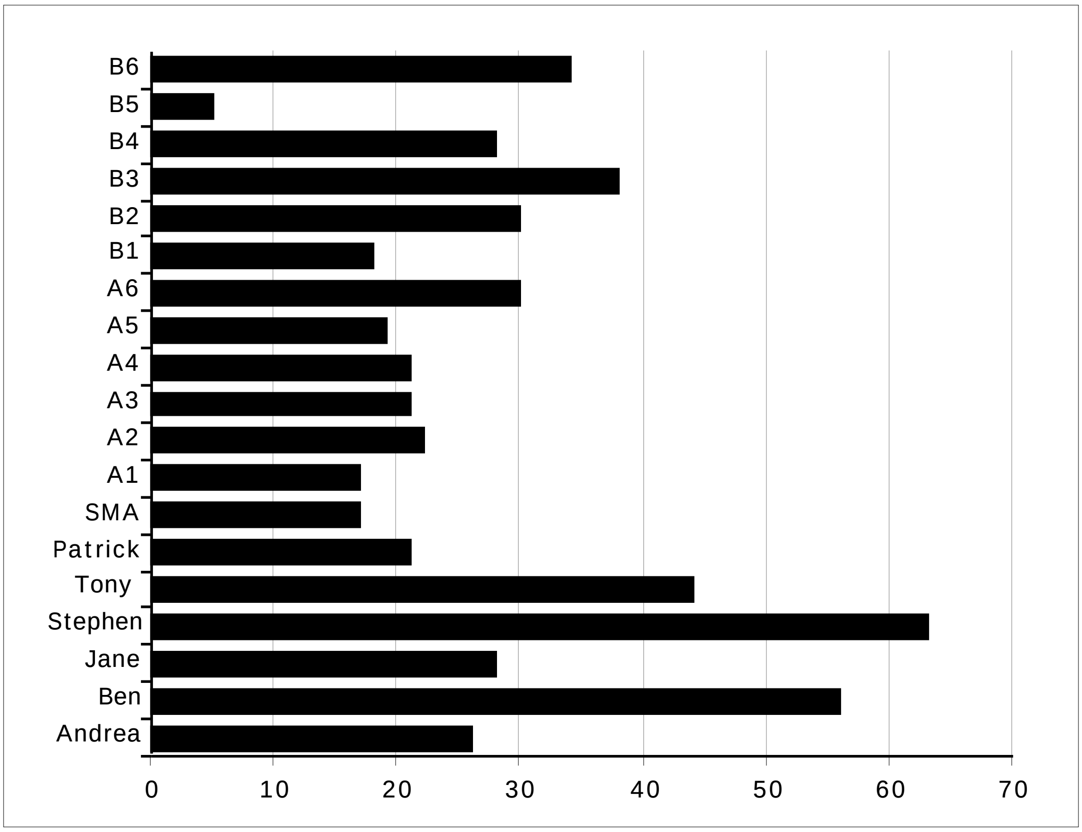
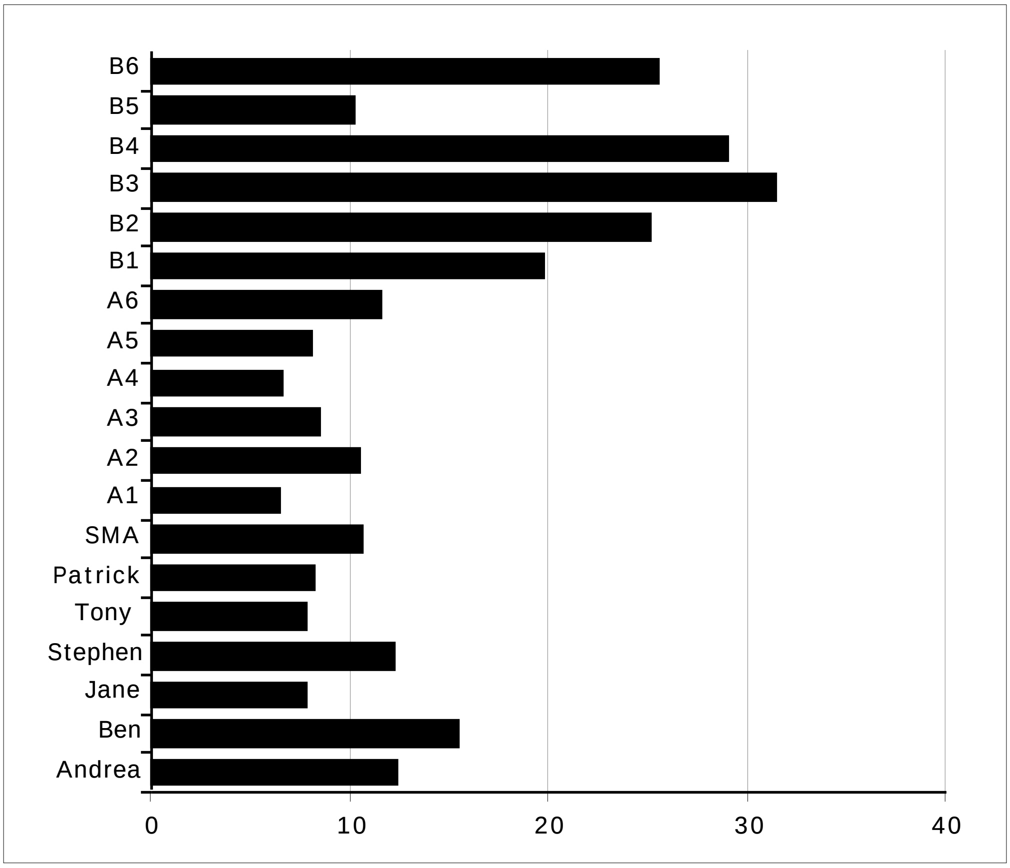
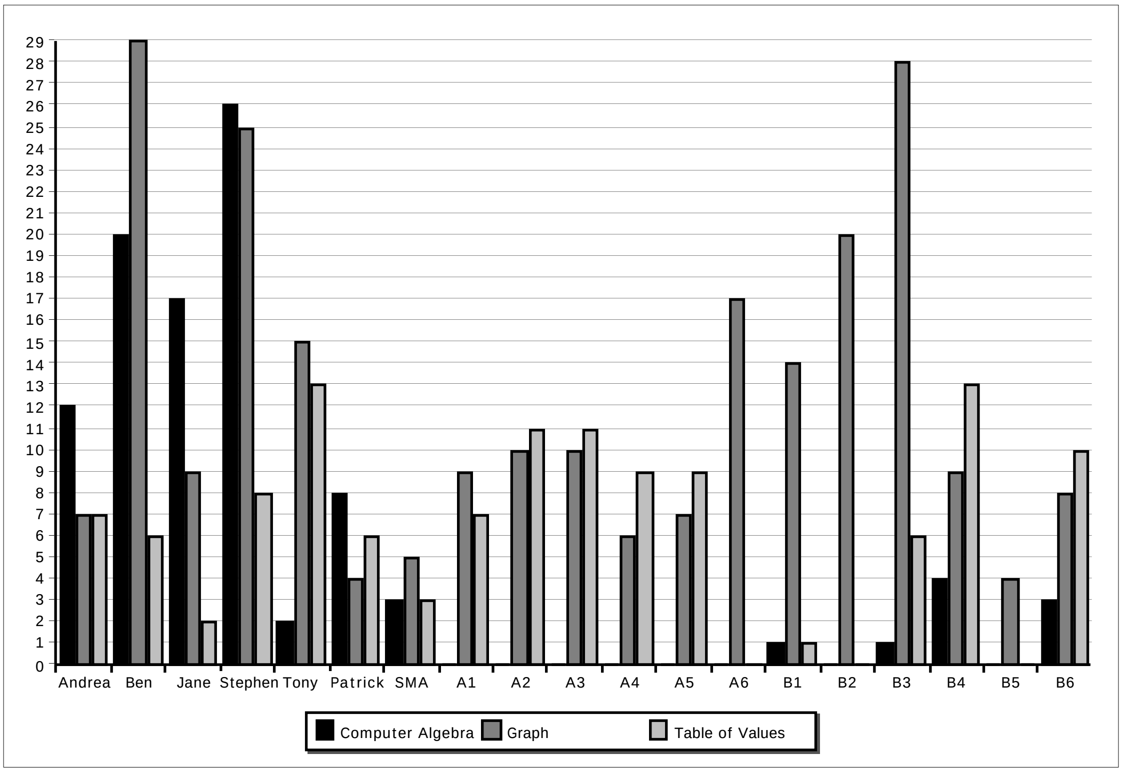


Figure 8.3: Relative Frequency of Tool Use across Participants (Percentage of total text units)



When this tool use is broken down into major software types (computer algebra, graph plotter and table of values), further detail emerges (Figure 8.4). The use of computer algebra tools among the Group B preservice teachers was minimal, even though this had been particularly emphasised in the introduction to the units of work (Group A did not have access to these tools). Among the students computer algebra use was far more common, but this occurred in the presence of the researcher who had made such use a particular priority for them. A similar emphasis upon the use of the table of values utility resulted in equally diverse results. For the students, the table of values was generally far less utilised than either graph plotter or computer algebra tool (although Andrea appeared to use graphs and tables equally) while preservice teachers appeared divided across the two groups - half showing some preference for the table of values over the graph plotter, and the rest displaying a strong preference for the graph plotter.

Figure 8.4: Frequency of Use of Tool Types



These varied patterns of usage across the participants will be examined in terms of two critical variables: the effects of mathematical **content** and **process**. The first considers the curricular context of the activity: whether participants were studying “Beginning Algebra” or “Calculus”, for example, and the nature of the tool use within this framework. The second considers the nature of the mathematical activities which accompanied and directed the use of available tools - which activities appeared to be most often associated with the use of computer tools? Additionally, the effects of the tool use will be examined through consideration of the respondents' own assessments - why they used the tools and how effective they found them to be.

The Curricular Context

Table 8.1 summarises the breakdown of tool use in relation to curricular content areas for the various participants. "A" represents the use of an "algebra" tool (*Theorist, MathMaster* or *CoCoA*), "G" represents use of a graph plotter, "T" stands for "table of values" and "√" indicates that the curricular topic was encountered but no software tools were used. It is clear from the table that the category labelled "Problems" was that most extensively encountered across all but the youngest participants. This grouping includes, not only those questions contained within the module, *Something to Think About*, but also open-ended and student-generated mathematical tasks which occurred frequently throughout the interactions. A clear focus for the group A preservice teachers was upon "early algebra" - the modules *Beginning Algebra* and *Equations and Problem Solving*, while the Group B preservice teachers were more engaged in problem-based activities. The breadth of curricular coverage by the various students may also be observed from the table.

Several features may be noted from this display. The use of the **table of values** was most common among the early algebra modules, where it represented an explicit priority. Similarly, the **graph plotter** was a central feature of the *Curve Sketching* module. Of the students, once again Andrea may be observed to give equal preference to the graphical and tabular representations; in contrast, Ben and Jane display a strong preference for the graphical representation, a preference shared by the

Group A preservice teachers, as evident in their approach to the mathematical problems they encountered. The Group B participants, however, appeared to make more effective use of the Table of Values in their problem-solving, but reverted to the graph plotter exclusively within the curricular contexts which they encountered.

Table 8.1

Tool Use and Curricular Content

	Beg. Alg	Eqns	Coord Geom	Curve Sketch	Comp the Sq	Inv. Fns	Calc.	Review	Probs.
Andrea (S1)	GT	GT	√	AGT				AGT	AGT
Ben (S2)		√	G	AG	G		AG	√	AGT
Jane (S3)	√	√	AG	AG	G			A	A
Stephen(S4)			√	AG	AGT	AGT	A	AT	AG
Tony (S5)	T	AGT		G				√	
Patrick (S6)	AGT	A							
SMA (T1)									AGT
A1	GT	T							G
A2	GT	GT							G
A3		GT							G
A4	T	GT						√	G
A5	GT	GT							G
A6				G					
B1					G		G		GT
B2							G		G
B3					AGT		AG		GT
B4	√						G		AGT
B5					G				G
B6		GT							AGT

The categories of **Problems** and **Reviews** provide the clearest insight into patterns of preferred tool use. Within several content-based modules, the majority of tool encounters were prompted explicitly. The focus within the *Equations* module, for example, was upon versatile thinking about equations, and users were prompted to access both graphs and tables specifically, and computer algebra if it was available. Within *Curve Sketching*, the graph plotter was the tool of choice. Use of other tools (as demonstrated by Andrea, for example) suggests initiative and cross-representational facility. Similarly, the use of tools within the *Review* modules was at the option of the user, and further demonstrates the extent to which the various participants had accepted the use of the computer tools as part of their mathematical learning experience. Ben, for example, worked through several review modules, and (although prompted at various points by the researcher) chose not to use any available tools. Andrea, on the other hand, displayed a willingness and, indeed, enthusiasm for the use of the tools which was unique among the participants. Although Stephen made effective use of a range of tools on various occasions, he displayed, like Ben, a reluctance to avail himself of their support, appearing to see it as fundamentally incompatible with his view of mathematics learning.

Ben notes, for example, that "I find most of maths OK - I can just bang it out, but when I don't know where something comes from or why it works I can't remember it". Ben was referring specifically in this case to rules such as the derivative of the natural logarithm of x being 1/x. He goes on to comment,

> Computers help me to visualise the question being asked. It also presents different methods to answering questions, e.g. if you didn't know the substitution or elimination method it can alter a question and make it easier to understand and easier to work out.

Ben attributes the active role in this encounter to the computer, which "presents" different methods and can "alter a question". In fact, he appears to be placing the computer tool in place of the teacher or tutor, who is expected to perform these functions, allowing him to adopt a passive role in his own learning. This passive role is further emphasised by Ben's reliance upon visualisation in his mathematics learning, in which he can simply "be shown" what to do and how to do it.

> Seeing the graph it verified my result - I am now 100% sure that 6 $^{2}/_{3}$ is the maximum and greatest speed is $^{4000}/_{27}$.
>
> [I used the table of values] to verify my results. Yes it was effective - it made me more confident of my answer... It was effective but I always find the graph more helpful - I like to visualise.
>
> The "functions of motion" model was helpful to visualise velocity over time - it's hard to picture when it's all just numbers.
>
> [I used the table of values] to compare two functions. It wasn't effective because I didn't pay any attention to it.

Ben used the computer primarily as a tool for **verification** of results which he had already obtained in most cases by traditional methods. While this is indeed a powerful role for algebra software tools, it is also an extremely limiting one. It places the learning emphasis firmly upon traditional methods, and the computer as an "optional extra". As a **purpose** of tool use, **verification** may be contrasted with **exploration**, with its implications of enquiry and student-initiated mathematical activity. Ben's comments suggest, too, a limitation of the tabular representation. While the graphical image is global and immediate, conveying information in an intuitive visual form, the table of values presents a relatively large amount of information in numerical form, which is more difficult to process and interpret. As Ben noted, if the user is not "paying attention", tabular information is easily overlooked. Ben's passive and visual approach to algebra learning would appear to work against effective use of tools such as the table of values, which require a more active analytical approach.

This distinction between **visual** and **analytical** use of computer tools appears significant in terms of understanding the preferences for different tools. While the graph plotter offers a fundamentally *visual* representation (in both the van Hiele sense and that of the SOLO *ikonic* mode of thinking), the table of values is essentially *analytical*, requiring quite complex information to be processed serially. This is not to imply that each tool cannot be used in the other mode. It was for this reason that the graph plotter developed for the study was enhanced to allow the coordinates of points to be displayed when the mouse is moved across the graph space. In particular, if a function has been graphed, the particular coordinates of the graph may be "traced" out using this feature, encouraging and supporting an analytical view of the functional information. Similarly, using the table of values to compare two functions allows an immediate visual response by the user as to whether the functions are identical or not. Stephen observes this feature when he notes:

> I used the table of values to compare two possible answers for a multiple answer question. It was effective as it showed the different results that each answer could obtain and how different the answers were.

In fact, for purposes of comparison, Stephen preferred the tabular representation to the graphical.

> I found the table of values most convincing; the graphs were helpful but not 100% sure ... To see what set of numbers were less than 5. Yes it was effective in doing this - probably more so than the graph since it just said true or false.

Stephen observed that the computer extended his mathematical competence but, like Ben, appears to value more his own ability to work unaided.

> We used the algebra tool to graph, differentiate and to find the minimum value of the question. This was effective as it showed the steps to use and in what order to find the answer. There were a couple of things I could not have done like factorising the large equations and finding the final answer in differentiating. To see the graph helped me to understand the answer better.

Jane, too, observed that the computer algebra tool extended her mathematical abilities.

> *[I used the computer algebra tool] to substitute into polynomials, solve simultaneous equations. [It] helped me to solve the problems that I couldn't do with pen and paper. I felt strange because I haven't used it before but I wouldn't mind using it again.*
>
> I used computer algebra for questions involving algebra and fractions and surds. We were simplifying and substituting and solving complicated equations. I found it to be helpful because it helped me to work out the questions. I think it will help me next time I get a question like that because I could see the steps involved and I feel I understand them better now.

Like Stephen, Jane points to another significant property of tool use in this context - the computer algebra tools made the mathematical process **explicit**. These were not programs which simply produced an answer; rather they involved the user in developing the solution, using their mathematical skills, and this feature was considered advantageous by the students. Ben and Stephen have already been noted as preferring to see **alternative approaches** to solutions, a further advantage perceived in the use of such tools (although subject to the intervention of the tutor).

Andrea reiterates several of these features of tool use when she notes that "we used [computer algebra] for recognising different types of equations and graphs, coordinate geometry and their uses. It was effective realising that there was more than one way to look at an equation".

At the same time, effective use of computer tools is limited by, among other things, the zone of proximal development of the user. The manipulative facility of computer tools is insufficient as a basis for understanding. As Patrick observed,

> When I used the computer to solve some equations I didn't learn anything because the computer did it for me, leaving me only with an answer, not the knowledge of how to do it.

Certain critical properties of tool use may be recognised at this stage. These include **depth** (visual or analytical), **purpose** (verification or exploration), **breadth** (the extent to which alternative approaches are available) and **process** (the extent to which the mathematical process is made explicit). Greater detail may be gained by consideration now of the relationship between tool use and mathematical activity.

The Mathematical Context

The most common mathematical actions associated with tool use involved either **representation** or **manipulation**. The latter involves algebraic activities traditionally performed using pen and paper, but now available using computer tools, such as *Theorist* and *MathMaster*. Ten such actions were identified as occurring most frequently:

- Simplify
- Substitute
- Solve linear equations
- Solve non-linear equations
- Solve simultaneous equations
- Expand
- Factor
- Evaluate
- Differentiate
- Integrate

Table 8.2

Tool Use and Mathematical Context

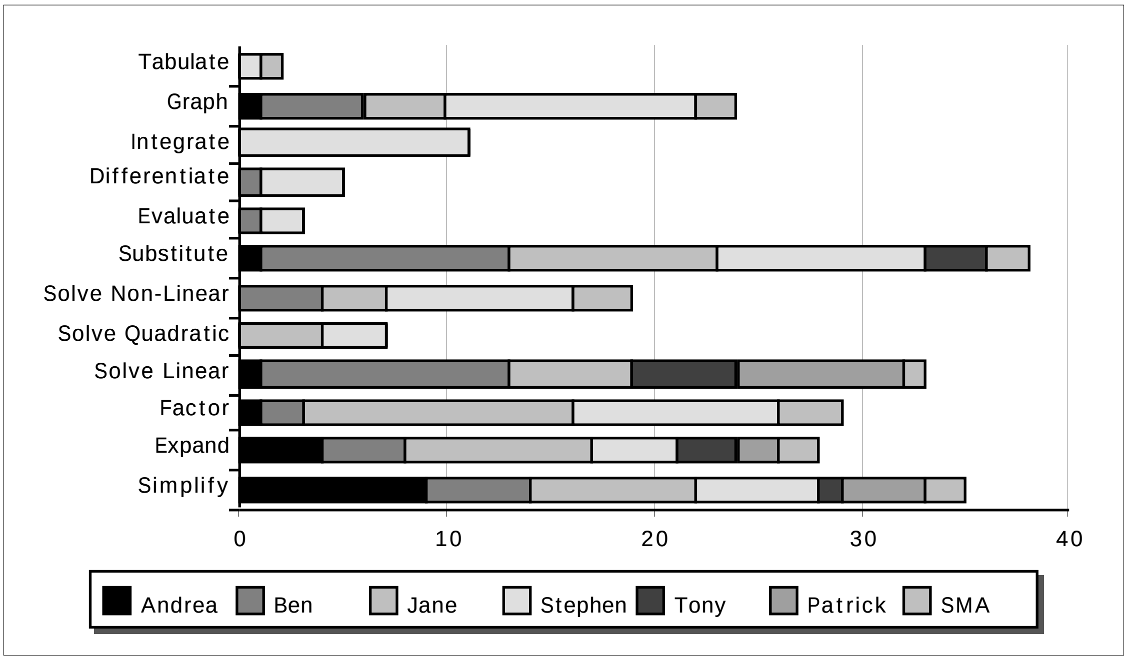
	Simplify	**Subst.**	**Solve Linear**	**Solve Non-L**	**Solve Simul.**	**Expand**	**Factor**	**Evaluate**	**Calculus**
Andrea (S1)	A	A	AT	A	A	A	A		
Ben (S2)	A	A	AT	A	AG	A	A	G	A
Jane (S3)	A	A	AGT	A	G	A	A		
Stephen(S4)	A	A	T	A		A	A		A
Tony (S5)	A	A	AT		T	A	A		
Patrick (S6)	A	A	A			A		G	
SMA (T1)	A	A		AGT		A	A		
A1			GT	GT				GT	
A2			GT					GT	
A3			GT					T	
A4			GT					T	
A5			GT					T	
A6									
B1									G
B2							G	G	G
B3				GT			G	AGT	G
B4				GT			A	A	
B5							G	G	
B6			T	G					

Table 8.2 displays the tool use by the various participants which accompanied these mathematical actions. The use of multiple representations for equation solving was common for linear equations (since it occurred most frequently within the curricular context of the instructional modules, which required such use), but rare for non-linear functions (which formed the basis for several of the problems encountered). Similarly, the use of the graph plotter as a tool for factorising and calculus by several of the Group B preservice teachers was indicative of a high level of mathematical and technological sophistication - they were using the tool in inventive and strategic ways.

This notion of **strategic** tool use has already been identified as central to the concerns of this study. Such use is defined to be deliberate, goal-directed and insightful, and may be recognised as frequently versatile (involving the use of a variety of representations), active, analytical and displaying a repertoire of available and appropriate mathematical strategies. While my own use of the mathematical tools might be expected to reflect characteristics of strategic use, the evidence of Tables 8.1 and 8.2 suggests that Andrea, Stephen and several of the Group B preservice teachers may well be considered in this light. In terms of the curricular context, it appears likely that strategic use of the software tools may be limited to open-ended problem-solving situations, where the user must display initiative and some measure of inventiveness, rather than those content-based activities where the path to follow is largely predetermined.

Figure 8.5 provides a detailed breakdown of the pattern of mathematical use by the students of the computer algebra tool, *Theorist*. Because of its simple and intuitive interface and broad mathematical functionality, this was the preferred algebra tool for the project, and that most frequently used by the students. The most frequent mathematical actions for which the students used computer algebra were substitution, simplification, solving linear equations and graphing, although the pattern was different for each participant. These results are not intended to generalise beyond this sample, dependent as they are upon the particular focus and tasks posed within the study. In this context, however, they provide a valuable overview of frequency of use of the various mathematical capabilities of the program.

Figure 8.5: Frequency of use of mathematical functions of *Theorist* by students and researcher



For Andrea, the principal computer algebra activities were simplifying and expanding, with much less frequent use for factorising, substituting, solving and graphing. Since she also made quite extensive use of both graph plotter and table of values, this suggests that she was not dependent upon the algebra tool, but used it to complement the other available tools. Consider the following excerpts from Andrea's text record:

> ************************
> Describe the difference between k*abs(x) and abs(kx) for constant k?
>
> For K*abs(x) the question is asking the absolute value of x the times by K but abs(kX) is asking for the absolute value of kx,. If K is a -ve number and was inside the function i.e. abs
> ************************
> Table : -4abs(x) : 5:05:24 PM
> 129 seconds
> ************************
> Comment: 5:07:55 PM
> For k*abs(x) and abs(kx): If k≥0 then k*abs(x)= abs(kx) if k<0 then k*abs(x) would be the exact reflection of abs(kx) i.e. abs(-4x)=4x but -4abs(x)=-4x => -abs(kx) = k*abs(x). Because k is negative, the RHS will be a negative graph.
> ************************

When confronted by a symbolic problem, Andrea turned to the table of values in order to consider it in numerical terms, which had more meaning for her than the symbols. By considering a single case (when k = 4) Andrea compared the two functions **4abs(x)** and **abs(4x)**, and then generalised from that. Although she used the table of values as her tool, she interpreted the tabular and symbolic results in graphical terms, as "reflection" and "negative graph". This is suggestive of strong cross-representational links. Her tutor suggested that she work through the section of the *Curve Sketching* module which dealt with the absolute value function in order to clarify some of her uncertainties in this regard.

At this point, Andrea encountered the definition of absolute value as **abs(x) = $\sqrt{(x^2)}$**. This was unfamiliar to her, and it was suggested that she investigate further. Her attention was also drawn to the possible distinction between **$\sqrt{(x^2)}$** and **$(\sqrt{x})^2$**, in light of her previous consideration of **k*abs(x) and abs(k*x)**. To this end, she chose the *MathPalette*, entered the equation **$\sqrt{(x^2)} = (\sqrt{x})^2$**, and engaged in actions clearly suggestive of strategic software use.

> * MathPalette© 12/9/94 4:36:32 PM
> Table: 4:42:17 PM
> 55 seconds
> Table: sqrt(x ^ (2)) = (sqrt(x)) ^ (2) : 4:42:19 PM
> ************************
> Comment: 4:43:54 PM
> The two functions are equal when x >= 0,and are not equal when x < 0.
> When x >=0 then the values for x, y1 and y2 are all the same.
> ************************
> function2 : √(x^2)
> function1 : [√(x)]^2
> ************************
> Comment: 4:46:25 PM
> So when you square a number and then square root it, it becomes the positive original number.
> ************************

Andrea offers an hypothesis, based upon her reflections upon the table of values result. She then turns to the graph plotter to further validate this conjecture.

HyperGraph : 4:48:00 PM
95 seconds
Function: (sqrt(x))^2
4:48:39 PM
****
Comment: 4:49:27 PM
So the absolute value of x is the squared square root: abs(x) = √(x^2)
true for all values of x,while [√(x)]^2 = abs(x) when x ≥ 0.
****
Comment: 4:51:48 PM
I expected √(x^2) to be a parabola, because from the table of values,
they were symmetrical.
****

Note here that Andrea's visual image derived from the table of values had been incorrect. She had been misled by the presence of the exponent to expect a parabolic shape. Viewing the graph corrected this misconception. While this could have involved simply a surface viewing of the graph, Andrea analysed the graphical result in light of the symbolic form and was able to appreciate why the shape of the graph was linear rather than parabolic.

At this stage, Andrea was asked to consider an example of an equation involving the absolute value function. She turned to the graph plotter and observed the graphs of the two functions which made up the equation. Unsure as to whether she had found all possible solutions, Andrea then selected the computer algebra tool and used it to solve the equation symbolically.

MathPalette4:55:46 PM
7 seconds
HyperGraph : abs(2* x - 3) = x + 5 : 4:56:50 PM
abs(2* x - 3) = x + 5
****
Comment: 4:58:43 PM
My confidence at this stage is about 90%. -2/3 I am 100% sure.
****
Algebra Tool 4:59:26 PM
43 seconds

```
Theorist  Student Edition :  abs(2* x - 3) = x + 5 : 4:59:26 PM

(2*x-3)^2=(x+5)^2
4*x^2-12*x+9=x^2+10*x+25
3*x^2-22*x-16=0
3*(x-8)*(x+2/3)=0

abs(2*x-3)=x+5
abs(2*x-3)^2=(x+5)^2
abs(2*x-3)^2=x^2+10*x+25
```

Her ability to apply the method of "squaring both sides" using the algebra tool was illustrative of an action which was probably beyond her capabilities to successfully attempt unaided. *Theorist* offered her the support to explore with confidence an algebraic process which was new to her. Andrea used the available tools with deliberation and clear intent. They provided both manipulative support and enhanced representational facility which resulted in what appeared to be improved understanding of the concepts involved.

Stephen tended to use the available software tools to support and verify his own computations. Having access to a computer at home, he was given a copy of *Derive* which he was encouraged to use. This occurred rarely, and then only to view the graphs of functions, believing that the mastery of the manipulative aspects of algebra was a requirement of his course, and that there was little to be gained in having a computer perform these. Although he used computer algebra software often in his regular tutorial sessions (at the prompting of his tutor), such use was seldom spontaneous. He appeared to view using the computer for algebraic manipulation as a form of "cheating", although he was willing to use it to verify his own solutions, in the same way that he used the answers in the back of a textbook.

One aspect of tool use which became increasingly significant for all students as the collection of data proceeded was the perception of **confidence** in their solutions. Stephen, Ben and the others demonstrated that they were willing to accept answers in which they had, at times, less than full confidence, even while computer tools were available by which these results could be verified. This aspect of tool use became a major area of focus in the later stages of data collection, since it has important implications for students perceptions of mathematics learning, and their own responsibilities in this regard. It will be considered in greater detail later in this chapter, but Stephen provides an example of his reluctance to use the technology, even when his own skills may be insufficient to guarantee a complete or even correct result. The example arises from attempts by Stephen to answer a problem posed within the module, *"Something to Think About"*. The problem points out that the equations, $\mathbf{x = 2}$ and $\mathbf{x - 2 = 0}$ are considered to be mathematically equivalent statements, and yet squaring one produces an equation with two distinct solutions ($\mathbf{x^2 = 4 \Rightarrow x = -2}$ **and 2**) while the other produces a single repeated root ($\mathbf{(x - 2)^2 = 0 \Rightarrow x = 2}$). Stephen appeared unable to come to terms with the requirements of this question - he appeared to expect a question which was "well-defined" with a clear "right or wrong" solution. Eventually (after prompting from his tutor), Stephen used the graph plotter to study the graphs of the different functions and equations but was unable to offer an explanation with which he was satisfied.

The second part of the problem involved the equation $\mathbf{x - \sqrt{6 - x} = 0}$. He approached this almost with relief, as it appeared to fit more closely with his perception of an "algebra problem" - it signalled to Stephen that he should initiate equation solving techniques, beginning with squaring to eliminate the radical. This method proved unsuitable and Stephen's attempt at solution faltered. Unable to proceed, it was suggested that he might "use the computer" to help. His response was to choose the graph plotter and, later, the table of values:

Card: x = 2 and x-2 = 0? : 6:17:02 PM
Button: Hypergraph
Grapher: 6:25:56 PM
****
Comment: 6:28:14 PM
This shows that the function crosses the x axis at 2. About 80% confidence in 2 as a solution.
****
198 seconds
Table: 6:29:19 PM
****
Confidence now at nearly 100%
****

Since he still expressed some hesitation regarding his solution, it was suggested that he might use computer algebra to assist with the manipulation (and, in particular, he might try moving the radical expression to the right-hand side of the equation prior to squaring):

Button : Computer Algebra
Theorist:
• x - √(6 - x) = 0
Move √ to RHS:
x = √(6 - x)
Square both sides:
x^2 = -x + 6
Move all to LHS:
x^2 + x - 6 = 0
Factor:
(x + 3)(x - 2) = 0
Solve for x:
x = 2 and x = -3
Substitute x = -3 into quadratic: 9 = 9
Substitute into original: -6 = 0.

****
Button : HyperGraph
Grapher: 6:33:05 PM
HyperGraph : x^2=6-x : 6:33:07 PM
135 seconds
****
Comment: 6:35:36 PM
I found the table of values most convincing, the graphs were helpful but not 100% sure.
****

Stephen verified his solution in two ways - first, by substituting both values derived from the quadratic back into the original equation (revealing that only one was a valid solution), and then again using the graph plotter. He nominates the table of values, however, as offering the most convincing evidence for the solution, while the algebra tool supported his manipulations. Although prompted towards the use of tools in this situation, Stephen’s actions (and, in particular, the multiple verification of his results) suggest strategic use.

Further evidence of Stephen’s reluctance to spontaneously use available tools (even when his own skills proved insufficient) is offered by a transcript of his attempt to complete the “Senior Algebra Review”. It was emphasised prior to this review that Stephen should use the available tools if he was unsure of an answer - in fact, he would be penalised for incorrect responses. Even so, it took an incorrect result and his own confidence dropping to 50% before he chose to use the *MathPalette* tools to check a response. Stephen’s hesitation regarding tool use is captured in the following comment:

> The down side of using computer tools is that in the test you can't use it, and you also learn the steps that you can do in the test. The steps that you have to do to get the answer to a particular type of question help you to get a feel for that type of question. When using the computer, you can see it doing it, but you don't think as much.
>
> On the plus side, using the computer has helped me by showing the easiest way to get to an answer and the setout of how to go about answering it.

Button : Senior Algebra Review
Card : Quiz 1: Question 2 : 4:52:49 PM
****
Comment: 4:53:35 PM
100%
****

For Question 2, Stephen indicates confidence of 100% in his solution.

Button : A*
Card : Quiz 1: Question 3 : 4:53:43 PM
CTRL-3 : 4:54:21 PM
****
Comment: 4:54:21 PM
90%
****

Although option "A" was the correct response, Stephen was only 90% confident.

Button : D*
Card : Quiz 1: Question 4 : 4:54:33 PM
CTRL-3 : 4:55:16 PM
****
Comment: 4:55:16 PM
70%
****

Again, Stephen chose the correct response (D) but his confidence was now only 70%.

Button : E
Button : C*
Card : Quiz 1: Question 5 : 4:59:37 PM
CTRL-3 : 5:02:09 PM
****
Comment: 5:02:09 PM
80%
****

His first error (for which he had assumed 80% confidence).

Button : B*
Card : Quiz 1: Question 6 : 5:03:00 PM
CTRL-3 : 5:05:48 PM
****
Comment: 5:05:48 PM
50%
****
* MathPalette 30/8/94 5:06:15 PM

His confidence now down to 50% for Question 6, Stephen selects the *MathPalette* as an aid to his substitution.

Although he possessed the mathematical competence and the knowledge and experience with the various tools to make use of them in a strategic way, it was Stephen's perception of the existing mathematics learning culture which proved the major impediment in his use of the technology in a spontaneous and practical way. The use of the software tools by Stephen varied for the most part between what might be termed **passive** (initiated and directed by the tutor) and **reflexive**, in which tools were chosen but used in a superficial way.

The fact that Stephen failed to fully engage the problem itself is also a significant factor in this context. If mathematics learning involves a tension between **enquiry** at one extreme and **instruction** at the other, then Stephen appears to identify most readily with the latter extreme. “Mathematical” problems are those which are “well-defined” and possess a clear and attainable solution. Exploration is perceived as a “luxury” which time in preparation for the Higher School Certificate examination does not allow. Stephen’s mathematical use of the computer tools available to him was dominantly representational (and strongly graphical in this regard); manipulative use was relatively rare, and restricted to those operations well beyond the scope of his abilities - large and difficult computations which were unlikely to be encountered within the limits of his examination preparation.

One further aspect of tool use which Stephen exposed in an early contact with a computer algebra tool (in this case, *Derive*) involved the capability of computer tools to make mathematical thinking *public*. The capability of the computer to make algebraic thinking explicit was an important element in the interview with Stephen which follows, in which the use of the computer algebra program was being demonstrated.

Friday, 7th May, 1993
(Stephen is given a few minutes to work on a solution to the question involving a cubic graph and its roots.)

I: (Gives the sheet with the graph of $f(x) = x^3 - 12x$ and four options to be judged true or false and sets the function up on the computer using DERIVE). Now the first question says that the maximum value of the function is 16. Your solution was . .

S: 65.

I: F(5) = 65 (from the graph) and that’s obviously correct. (Enters F(5) on DERIVE) We put in F(5) and press S for simplify . . and out comes 65. Why did you use that? Um, why did you say that?

S: Um, because the maximum value is the positive in the x (indicates the given domain for the function, $-5 \le x \le 5$). And if it only goes . . if the range is between 5, you put in the 5 . . into the x values, and that will get you the maximum value on the y axis.

I: Right. So this is only in this sort of case, where it's a restricted domain?
S: Yes.
I: Is it always the maximum positive x value that gives you the maximum value of the function? Say you had a different graph. Would it be possible to have one where the maximum occurred at -5 instead of plus 5?
S: I don't think so.
I: So even if you had a graph that was the reverse of this one . .
S: Yeah . .
I: So the maximum occurs at the highest point? Okay, so we weren't tempted by that there (indicates the relative maximum on the curve) . It's obviously not the maximum is it?
S: No.
I: Okay, what if . . it wasn't a restricted domain? What if it didn't start at -5 and end at 5, it took all values? What would you say then about the maximum value of the function?
S: Um . . it's all reals on the y axis . .
I: But if we wanted the maximum value? Someone said, "What's the biggest . . the maximum value of the function?" What would you say?
S: (Pause) uh, you wouldn't be able to give one, because . .
I: It hasn't got one has it?
S: It's infinity.
I: That's right. Good, okay. . very good. Now next. It says, 'The equation f(x) = 0'. Now, from what we were just talking about, so that's an equation there . . f of x equals nought. 'The equation has exactly two roots, one negative and one positive.' And your answer?
S: No.
I: What do you mean by that - 'it's situated at zero'?
S: Um, because zero is zero on the axis, it's . . they can't have a negative or positive value.
I: Alright. So how many roots does this equation have?
S: Just positive.
I: So when you let f(x) = 0, it has . . what . . just one root at zero?
S: Yeah.
I: What about these two? What do you understand by a 'root"', when it's asking for the 'root' of an equation?
S: Oh, was that for this part (indicates the function in question)?
I: Oh, it refers to the same function, yes, sorry. f(x) is still x cubed minus 12 x. Let's see what it says here (types on the computer).. This time, we will just take f(x) = 0, and create an equation. And notice, because it's an equation, you don't have to use the dots in front: you just use an equals sign. The other one was defining a function, so the computer thinks there is a difference between functions and equations. It treats them differently. f(x) = 0. If we asked it to solve - you see the 'soLve' command, you just type 'L' to solve. Solve expression number 4. It says x = 0 is a solution. So is minus 2 root 3, and so is plus 2 root three. So that gives three . . three roots. So when we are looking at the roots of an equation, we are looking at . . what in terms of the graph?
S: The places it cuts through the x axis.
I: That's right. So in this case, the equation, f(x) = 0, has exactly . . how many roots?
S: Three.
I: Three roots. So that is false. You were right that it was false, but for the wrong reason.
S: Yeah, I didn't know . . I didn't know that was for this question.

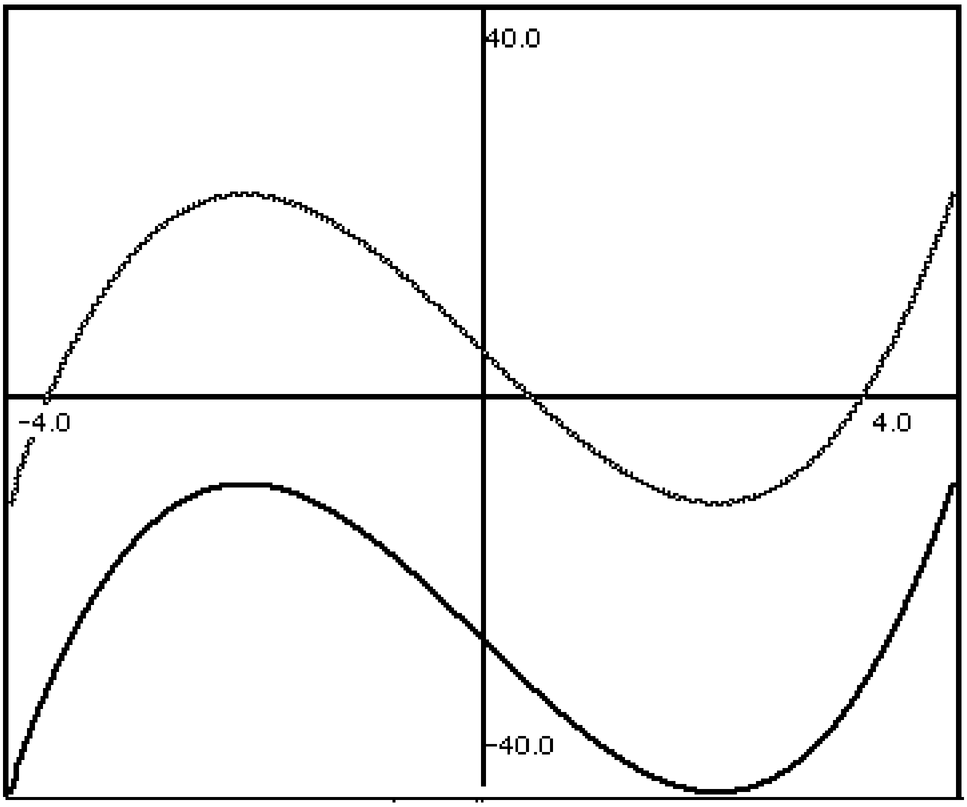
The computer algebra program is being used as a tool for demonstration and verification. At the same time, it is visually exposing elements of Stephen's thinking which may have been incorrect. While he displayed some uncertainty regarding the nature of the question, his understanding of key terms such as "roots of an equation" is made public in this encounter, and the tutor is in a better position to judge areas of which the student may have been unsure. The interview continued, exposing further gaps in Stephen's understanding.

I: Yes, well that explains it. Alright. What about part C: it looks a bit mysterious. 'f(x) = k has three real roots for all real k.' What do you think now about this answer, now that you know that it's talking about this thing here (indicates the function) ?
S: Um . . yeah, I think it has got three roots the same as the other one.
I: For all real k?
S: Yeah.
I: What do you think the question is talking about when it's saying, 'f(x) = k for all real k'?
S: Um, if you put any number in for k, like, how many times will it touch the x axis?
I: Let's ask the computer (types in f(x) = 2). If we put in f(x) equals 2: Now remember, we just solved it for f(x) = 0, which is that picture there (indicates graph) where it cuts the x axis. Let me put in f(x) = 2, and solve . . takes a bit longer this one . . Now we don't have to get tied up in what they actually are, just how many of them there are ... and there's number three. Okay, there are three roots. They are pretty horrible looking things, but we could tell it . . just as an example, if I . . see the last command is 'approX': if you type 'x', it approximates, and so instead of giving it in that exact form, it will spit it out as a decimal. It turns out that one is actually equal to about three and a half. So each of those is just a decimal number. What would they be representing, those three numbers?
S: ... Where it crosses the x axis?
I: Where what crosses the x axis?
S: The curve.
I: Which curve?
S: This one (indicates the graph of f(x)).
I: But this is f of x. Here we are talking about f of x equals 2 in this case.
S: So, if you put . .
I: Go on.
S: If you . . make x cubed minus 12x equals 2 . .
I: Good, go on.
S: And then you factorise it . . you 'take the two over'.
I: Right, take the two across . . It's now equal to nought . . so it's now more like what we are used to, isn't it? So we could graph this thing. Let's do that. What you have done there is f(x) minus 2 . . right? And we know this is equal to x cubed minus 12x minus 2 . . now if we plot that . . we might actually zoom out a couple of times before we start... to make it bigger. See that little cross there: that's at (1, 1). So it's giving you an idea of the perspective of the graph. Right, so, now. if we press 'P' for 'Plot' it's going to plot f(x) - 2, or x cubed minus 12x minus 2 . . Now that's a function. We had probably better call it something else . . let's call it 'G'. As a function, it has a graph, which we will see in about ten minutes, when the thing gets around to doing it, right? And it's actually equal to another function, right? It's equal to f of x, take away two. Okay, what we are interested in is this

here, which is not a function, it is an equation. Right? By your definition, which is quite right. So what does this mean, in terms of the graph that will eventually appear up there?
S: It will cross through the y axis at negative two . .
I: Yes.
S: And . .
I: How will you solve the equation by looking at the graph?
S: Um . . put the x values in?
I: Okay, there is the graph . . now where are the solutions to the equation?
S: Um . . where it hits the x axis at (0, 0), and at the other two?
I: At the other two, yeah. It's not actually (0, 0), although we've zoomed so far that it looks like it. It's actually . . what it should be, of course, is . . this picture (f(x)) moved down two steps. Right?
S: Mm.
I: And as we saw, it's still got three solutions. So your theory is looking good so far, that we will always have three real solutions. Okay, we used k = 2 as an example. Let's try something different . . give me another number?
S: Minus 5.
I: Okay, so we are interested in f(x) = -5. To graph that, what would you graph?
S: x cubed minus 12x plus 5.
I: Okay, so it would be f(x) + 5. The same picture as . . that equation . . plot . . Now if using k = 2 moved the graph down two steps, what will you expect k = -5 will do?
S: Move it up.
I: Move it up 5 steps. Alright, while it's thinking about drawing the graph (it could take a while) . . so the values of k have what effect upon the graph of f(x)?
S: Move it up or down along the y axis.
I: Alright. Then we will ask the question again: do you think it's true that, for any value of k, it will always have three real roots?
S: Yeah.

This interview followed soon after Stephen had offered his definitions of function and equation, described previously. His confusion regarding the distinction between the two and his association with the visual format had prompted the interview focus upon these two terms. The interviewer is careful to point out examples of each, and to use the terms correctly in order to emphasise the similarities and differences between them. Stephen demonstrates good skills of graphical interpretation, recognising the effects of adding and subtracting a constant term upon the vertical position of a graph. He is, as yet, unable to visualise that certain vertical translations will result in the cubic graph cutting the x axis in less than three places (Figure 8.6).

Figure 8.6: Comparing f(x) and f(x) - 30



I: Alright. So it wouldn't matter if you put in . . k equals thirty? What would k = 30 do to the graph?
S: Move it up thirty . .
I: Oh, remember, positive numbers actually moved it down, and negatives moved it up. So k = 30 will move it down 30 steps.
S: Can I see . . (inspects the original graph closely).

Note that Stephen is analysing the graph at this point, alert to the fact that his previous assumptions are being questioned. He now realises that the graph may cross the axis at less than three points, and that, as a result, the equation may have less than three solutions.

I: f(x) minus 30 (enters on the computer) . . and because it takes so long to graph it, what we will do is solve it instead: the same effect. It's thinking about it . . like all good students, it doesn't rush into these things. So what's happening here is . . it's taking f of x (that picture there) and moving it down thirty steps. How many solutions is it going to have? How many places where it will cross the x axis? ...So what it means is that, as you saw, when you move it down far enough, this "hump" is going to be below the x axis and it will only cross in a single place, and only have one root. So it means that there are times when it doesn't have three roots. Will it always have at least one real root no matter how

far you move it up or down?
S: Yes.
I: Good, well at least it will if the graph continues forever. If we are looking at this particular graph, how far would you have to move it down so that there will be no real roots?
S: 65.
I: Yes, more than sixty five. Alright, we can see that it has three real roots for some values, one real root for other values, would there be anywhere where it could have two?
S: . . . (Long pause)
I: Can't imagine any? Imagine that we move this graph gradually down. Alright, as you move it one step down, when k equals minus one . . I'm sorry, k = 1, and 2, we move it two steps down; three . . all this time, it's still got three zeros, three roots. What about when k equals 16? . . What will happen at that point? And I'll point out that that's actually the height of that . .
S: It will just touch the x axis at this one . .
I: So it will have . . ?
S: Two.
I: So there is at least one place where there are two solutions. Anymore? Anywhere else that you could imagine it would have just two?
S: Move it up 16.
I: Yeah, that's right. Okay. We've got the picture, and eventually this will have it drawn . . What it's drawing there, you remember, is f(x) minus 2, when k equals two, so it moves it down two steps. Now, . . we will let it go. Okay, you're doing well. Last question then. The minimum value of the function is minus 65. What did you say?
S: Yes.
I: Yes? No problem then. It's true, isn't it? What they were looking for, of course, there, was for people to be distracted by these . . what are called "relative maxima" and "relative minima". It means that, in that little area there, it's certainly a minimum point, but over the domain, it's not an absolute minimum. Okay. You did well.

The computer served a vital role in exposing aspects of Stephen's thinking about functions, equations and their links with graphs. He found the experience valuable and continued to relate strongly to the graphical form. His use of the manipulative tools available, however, remained minimal.

While Stephen was almost stubborn in his refusal to seek computer-based assistance for his mathematical difficulties, the same could not be said of Ben. As a student, Ben preferred the role of "passenger" rather than "driver", rarely exhibiting initiative and accepting responsibility for his own learning. His preference for "visualisation" reflects his desire "to be shown" and corroborates his generally passive approach to mathematics learning. Like Stephen, Ben had extensive opportunities over a prolonged period to become familiar with the software tools. Like Stephen, Ben rarely chose to use the tools spontaneously, even when relatively uncertain of his own response. Unlike Stephen, Ben was not unwilling to use the tools, but rather lacking the initiative to actively select them.

Consider Ben's approach to the problem: "What is the value of c if the vertex of the parabola $y = x^2 - 8x + c$ is a point of the x-axis?"

> Firstly, I graphed the parabola y = x^2 - 8x minus the value of c. Then I saw the vertex of this solution as (4, -16). Then I saw for the vertex to be a point on the x axis the y value would have to equal 0, so the vertex would have to be (4, 0). So I had to move the y value up 16 values thus making the y intercept 16. Therefore the equation would equal y = x^2 - 8x + 16 showing the value of c to equal 16. The table of values was also used to see if these values were correct.

Unsure of how to begin, Ben was prompted by the tutor to consider the graph of $y = x^2 - 8x$. This was followed by the process described above, demonstrating that Ben had competent mastery of the available tools (he actually used graph plotter, table of values and, finally, *Theorist* to check the factored form of his answer). His description is replete with elements of visual imagery, used to advantage when he imagined moving the entire graphical object up by 16 units to obtain his answer.

Ben was happy to use computer tools whenever prompted, both to verify and to obtain solutions to problems. While Stephen used computer algebra only for those manipulations which were beyond him, Ben used it as a convenience, especially for tiresome and routine manipulations such as the solving of simultaneous equations, and even for solving linear equations (which he was quite capable of solving "by hand").

The use of the available tools by the two younger students was, not surprisingly, far more limited mathematically than that by the older participants. Tony used *Theorist* in particular as his preferred tool, enjoying the support it offered him for solving linear equations (which he had recently encountered at school). Like Andrea, Tony was willing to use such tools spontaneously to both verify his work and to support his computations. His attitude towards the computer algebra program is captured in a short transcript in which he is demonstrating its use to a peer (C):

T:	Oh goody, it's Theorist. I'd better tell you about this [Looks through files]. This one looks good - it says 'Good intro'. See, it can do this kind of thing.
C:	It looks a bit hard.
T:	Sorry, it just looks hard. You can type in all kinds of stuff like this - like 'k 2a outside 3 times 4 times 999, 456a then outside brackets to the square root of 999, 456a and now we'll try and solve it. We are really testing it here.
C:	Simplify?
T:	Simplify? OK. Let's simplify this one.
C:	It probably can't be simplified.
T:	You're right. We can't do that one either, so we'll close off this one. What did we just do? Oh, it's alright now. Let's do one of these - is that pretty or what? [referring to a three-dimensional graph].

Tony demonstrates, not only a very positive attitude towards the computer algebra program, but a willingness to use it to "play" with mathematics which would traditionally be beyond his capabilities. This element of **curiosity** was also evident in Tony's interest in the extension module on *Chaos Theory*, He chose to work through this module himself, and demonstrated what might be considered strategic use of tools in his exploration, moving from numerical values to graph to explore features of interest which resulted from different values of the variable "r".

```
* Tony ( ) Exploring Algebra session : 4:00 PM, Tue, 1 Nov 1994
Card : Exploring Chaos  :   4:00:40 PM
Card : The sounds of chaos  :   4:00:45 PM
Button :  CHAOS [repeated 9 times]
Button :  Plot
Button :  The sounds of chaos
Button :  Plot
Button :  CHAOS
Field : r-values  :  -1
Button :  CHAOS
Field : r-values  :  4
Button :  CHAOS [repeated 8 times]
Field : r-values  :  1000
Button :  CHAOS
Field : r-values  :  1000
Button :  CHAOS
Field : r-values  :  100
Button :  CHAOS
Field : r-values  :  10
Button :  CHAOS
Field : r-values  :  11
Button :  CHAOS
* Tony 1/11/94 4:03:12 PM
---------------------------------------------
```

Since Patrick had not studied any algebra at school at the time of the data collection, symbolic manipulation for him was a meaningless exercise. Although he was successfully taught to use *Theorist* to solve linear equations by moving terms, he "didn't learn anything because the computer did it for me leaving me only with an answer not the knowledge of how to do it". Patrick was far more positive after using the concrete algebra facilities offered within the *Mathpalette*, and after engaging in the *Think of a Number* game within the *Beginning Algebra* module. This activity engaged the user in the use of variables as generalisers, playing the traditional algebra game (*I think of a number, multiply by 2, subtract 5...*) using the table of values representation to operate, not upon a single number or variable, but upon a listing of several numerical values. In this way, the student sees that the process of acting upon a symbolic variable corresponds to actions upon a potentially infinite array of numerical values. After engaging in this numerically based exercise twice, the student then attempts the same process using symbols within a computer algebra tool. In Patrick's case, the preferred tool was *MathMaster*, since the interface allowed the user to select an expression and then explicitly operate upon it using the four basic operations. To play the "number game", then, Patrick entered "n" and "2" and selected the multiplication symbol to produce "2n". He then entered "5" and chose "subtract" to produce "2n-5". Continuing the process eventually leads back to the original value, "n".

Patrick found this symbolic manipulation to be a meaningful experience after having grounded the procedure in numerical values, using the table of values representation. He demonstrated strategic use of the computer algebra tool, then, by developing his own *Think of a Number* game, using *MathMaster* to support his simplification. Once again, the algebra tool supported learning beyond that which would normally have been possible for this student, and contributed to both increased understanding of the nature and meaning of variables (as demonstrated in his definition), and also to improved skills of algebraic manipulation.

The students generally appeared to use the manipulative tools offered by the computer for purposes of verification and for convenience. The high incidence of acts of substitution reflects the ease which this mathematical operation could be carried out using the computer algebra tool, *Theorist.* Students had simply to "drag" the expression or value to be substituted onto the algebraic equation, and the result would be displayed. Similarly, the solving of equations was facilitated by the *Theorist* interface, allowing the user to "drag" terms across the "equals sign" in a physical simulation of that method of equation solving. The obvious preference displayed for this method of equation solving by the student participants reflects the fact that the software encouraged such an approach. Having access to such a facility appeared in no way to detract from their skills when working without the support of the computer tools.

A final feature of computer use which appeared significant for several of the students was that associated with entry of algebraic expressions and equations, and even of numerical expressions to be evaluated. Initial entry of algebraic forms into a computer application appears to be associated with a process of **reconstruction**, in which the user must **transfer** the visual stimulus of the algebraic expression or equation into a form which the software may act upon. This involves a deliberate consideration of the various components and their relationships which may act to move the cognitive level from a superficial visual mode to one which is more analytical. It was as a means of exposing such thinking that a palette was created by which algebraic forms could be entered, and which would make explicit the user's recognition of the various parts of which they are composed.

Students who were most competent in their algebraic skills appeared to have little problem entering algebraic forms using a variety of forms - the expanded text-format of several of the programs ("3*x^2-4*x+2), the simplified text-format developed for the *MathPalette* and instructional modules ($3x^2 - 4x + 2$, where the exponent was placed by simply using the option key with the desired number), or the point-and-click interface offered by the palette. Most participants found that the use of the palette was too slow and cumbersome, and preferred the simplified format, adapted from that offered by the programs *MathMaster* and *CoCoA*. All students demonstrated skill and familiarity with the use of their own calculators, which they were required to use in their mathematics classes. It was noted, however, that the entry of expressions in correct two-dimensional format offered by the *MathPalette* encouraged greater confidence in their solutions, since they could see all the components of complex numerical expressions. When evaluating the expression $\frac{2.5\sqrt{31.6}}{24.9}$, for example, Ben worked first with his calculator, and then entered and evaluated it using the *MathPalette*.

> Confidence in my first answer was about 80% because I am using my own calculator. Confidence now 100% because I saw the actual expression set out for me.

Ben observed that, when using a calculator, each term of the expression disappears after entry to be replaced by the subsequent term. The computer allowed the entire expression to be viewed, and so any errors of entry would have been visible.

Jane offers a response which may be generalised to each of the student participants regarding use of the software tools when she comments:

> They help you to get the answer. They show you how to work out a question. Out of computer algebra, graph plotter and table of values, I have found the graph plotter to be most helpful - it shows you what the graph might look like, If I had a program like the algebra program at home, I might use it, but more for the 2 unit type questions.

Jane's reference to "2 unit type questions" implies questions involving a high level of manipulative difficulty. It appears that the students in general found the most valuable aspect of using manipulative algebra tools to be in making explicit the solution process: "seeing the steps". The same could be said of a written solution, or of watching the tutor demonstrate the development of such a solution. However, the computer appears to offer one important advantage in this regard: it supports the students themselves in developing the steps, and so encourages their involvement as active participants in the process, rather than as spectators.

The mathematical use of available tools by the preservice teachers was quite different for the two groups. Nowhere was the distinction between **instruction** and **enquiry** made more explicit than in the involvement of these participants. As already noted, the Group A preservice teachers had chosen, almost entirely, to work through the modules associated with *Beginning Algebra*. The assessment requirements for this group explicitly rewarded their comments regarding the units of work and the tools. The result was that, in general, these participants engaged in the project as an evaluation exercise and their thinking and responses were predominantly pedagogical, rather than mathematical. The two occasions when Group A individuals encountered the problems in the module *Something to Think About*, their tool use consisted simply of viewing the graphs for the functions they encountered. Their use of the tools might be thought of as **random** as, clicking to move from page to page within the modules, they clicked a button here or there to view graphs which interested them. They then moved on without attempting to engage the question, in the same way that Stephen avoided confronting a question which did not appear amenable to a well-defined solution. The Group A preservice teachers worked through the units as required, dutifully using the tools required of them and observed positive and negative features of the program and accompanying tools, but at no point did they display mathematical interest or **curiosity** which would have provided a stimulus to explore the mathematical ideas which they encountered. They appeared to think like **mathematics teachers** rather than like **mathematicians**.

It was common for participants in Group A to describe problems which they encountered with both program and tools. The former were associated most often with perceptions of disjointedness and repetition in the activities for the different modules:

> At this stage I would like to comment on some of the [negative] features of the package. The setup of the package seems to be very disjointed. It does not flow from one item to another ... Some of the tools did not link in very well with the package... (A1)
>
> The graph facility was very time-consuming in the 'zooming out' process. The establishment of the point of intersection of the two graphs is quite hard to determine. (A3)
>
> The first module I worked through was Curve Sketching. I found it to be very useful, but it was also slow, repetitive and time consuming. (A6)

Although the use of the tools had been demonstrated for these students, it was common for them to encounter problems entering expressions using the palette, and using text-file entry methods. At the time that this data was gathered, the modules required algebraic expressions to be entered in "expanded" form - as "y = 4*x^2 - 3*x", for example, instead of the simplified form. In spite of explanation to this effect, several of the preservice teachers attempted to enter equations such as "y = 4x".

The problems encountered by this group in particular served to inform the continued development of the instructional modules and the accompanying tools, especially the improved interface allowing simpler and more intuitive entry of mathematical forms. Steps were taken to provide better links between the activities, and to provide easier navigation through the modules (these included the addition of a new menu from which users could move to any part of the program at any time, and so reducing the problems of "getting lost" within the modules).

The overall assessment of the program by this group was quite positive, allowing for the difficulties which they expressed. They remained enthusiastic regarding the potential of the computer as a tool for algebra learning.

> The use of the computer in the classroom is a way of encouraging and motivating students to learn algebra without doing all the textbook problems that are typically set in formal classes. (A2)

> Two areas that I thought the computer could help greatly, especially as a time saving device, was in the area of graphing functions and the area of tabling data. (A4)

> ...I see the main role of computer technology in supporting the teaching and learning of algebra as taking away much of the boring, unnecessary manipulative work that traditional approaches give to the students. In this way, they are promoting understanding and meaning rather than blind application of teacher imposed rules. (A6)

Other positive features recognised included the provision of meaningful context for algebraic ideas, the freeing of students to work at their own pace and the encouragement of discussion and group approaches. However, the Group A preservice teachers appear to concur with the students who perceive the principal advantages offered by computer tools as those associated with **motivation**, **convenience** and **representation**, rather than supporting and extending the traditional mathematical processes associated with manipulation of algebraic forms.

The evaluative comments of the Group A participants, then, proved highly valuable in terms of the ongoing development of the package and tools, and their perceptions were informative of the role of computer tools in algebra learning. It was clear, however, that this group had not engaged in the use of the computer as a *mathematical* tool. It was for this purpose that the Group B preservice teachers were included in the study and expressly encouraged to make problem solving a priority. The resulting incidents of tool use were varied, inventive and frequently strategic in nature.

Consider, for example, the attempts by B3 to answer the question from the module *Something to Think About* , involving the equation:

$$\left(x^2 - 5x + 5\right)^{x^2 - 9x + 20} = 1$$

Her first attempt involved repeated use of the graph plotter:

```
Button : (x^2-5x+5)^(x^2-9x+20)=1
y=(x^2-5*x+5)^(x^2-9*x+20)-1
Grapher: 11:56:50 AM
HyperGraph : y=(x^2-5*x+5)^(x^2-9*x+20)-1 : 11:56:50 AM
252 seconds
y=(x^2-5*x+5)^(x^2-9*x+20)-1
27 seconds
Button : (x^2-5x+5)^(x^2-9x+20)=1
y=(x^2-5*x+5)^(x^2-9*x+20)-1
Grapher: 11:57:23 AM
HyperGraph : y=(x^2-5*x+5)^(x^2-9*x+20)-1 : 11:57:23 AM
y=(x^2-5*x+5)^(x^2-9*x+20)-1
10 seconds
* MathPalette 16/9/94 11:57:38 AM
82 seconds
Grapher: 11:59:15 AM
HyperGraph : y=(x^2-5*x+5)^(x^2-9*x+20)-1 : 11:59:15 AM
y=(x^2-5*x+5)^(x^2-9*x+20)-1
41 seconds
* MathPalette 16/9/94 12:00:01 PM
HyperGraph : 12:00:03 PM
Grapher: 12:02:03 PM
HyperGraph : y=(x^2-5*x+5)^(x^2-9*x+20)-1 : 12:02:03 PM
y=(x^2-5*x+5)^(x^2-9*x+20)-1
11 seconds
* MathPalette 16/9/94 12:02:20 PM
```

This provides an example of what has been described previously as **reflexive** use of the software tools. Such use appears to have been encouraged by the facility within the instructional modules which allowed functions to be graphed by simply “clicking” on them. This appeared to induce at times almost automatic viewing of the graphs in a clearly **visual** way. Obviously frustrated, B3 moved on to other activities, but demonstrated persistence and determination when she returned to this question at a later time with an improved repertoire of tools, including the table of values and the inbuilt "solver":

```
* MathPalette 23/9/94 12:14:34 PM
Solver: Searching for the roots of  (x^2-5x+5)^(x^2-9x+20)-1 from -10 to
10 ...
5 solutions of (x^2-5x+5)^(x^2-9x+20)-1 have been found at x = 1 ,2 ,3 ,4
,5
HyperGraph : 5 solutions of (x^2-5x+5)^(x^2-9x+20)-1 have been found at x
= 1 ,2 ,3 ,4 ,5  : 12:31:58 PM
27 seconds
* MathPalette 23/9/94 12:32:30 PM
Table: 12:32:36 PM
x = 1 ,2 ,3 ,4 ,5  : 12:32:36 PM
Table :   :   12:36:00 PM
Table : 5 solutions of (x^2-5x+5)^(x^2-9x+20)-1 have been found at x = 1
,2 ,3 ,4 ,5   :   12:39:53 PM
************************
Comment: 12:40:16 PM
this is the new revised comment to this silly question. we have found 5
solutions the original 1, 4 &5. but now we have found 2 more solutions.
These solutions occur  when we have (-1)^2 and this is when x=3 (by
factorising both the base and index
************************
Comment: 12:46:47 PM
the other solution is when we have -1 to the power of a positive  which
occurs when x=2.
************************
```

This extract offers what might be considered a definitive example of strategic software use. It is deliberate, goal-directed, persistent and insightful, making thoughtful use of the range of available and appropriate tools to not only derive a solution, but to verify this result using multiple sources. It was included as an example of a problem which was not amenable to graphical solution (Figure 8.7), and yet was immediately accessible using the table of values. Those participants who were restricted to the graphical representation were disadvantaged in such a case.

Figure 8.7: A difficult function to graph

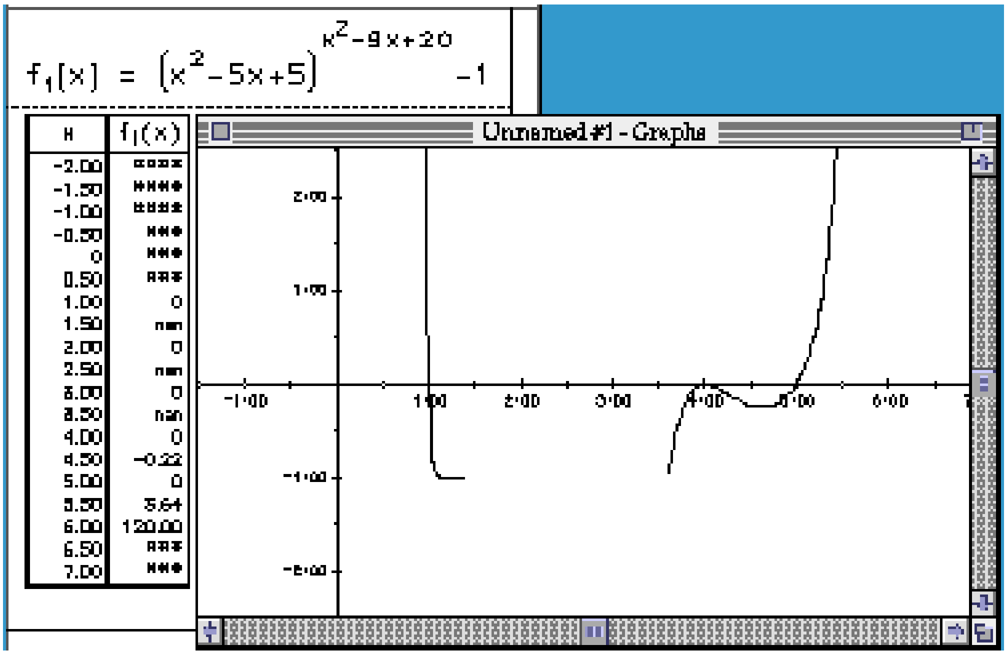
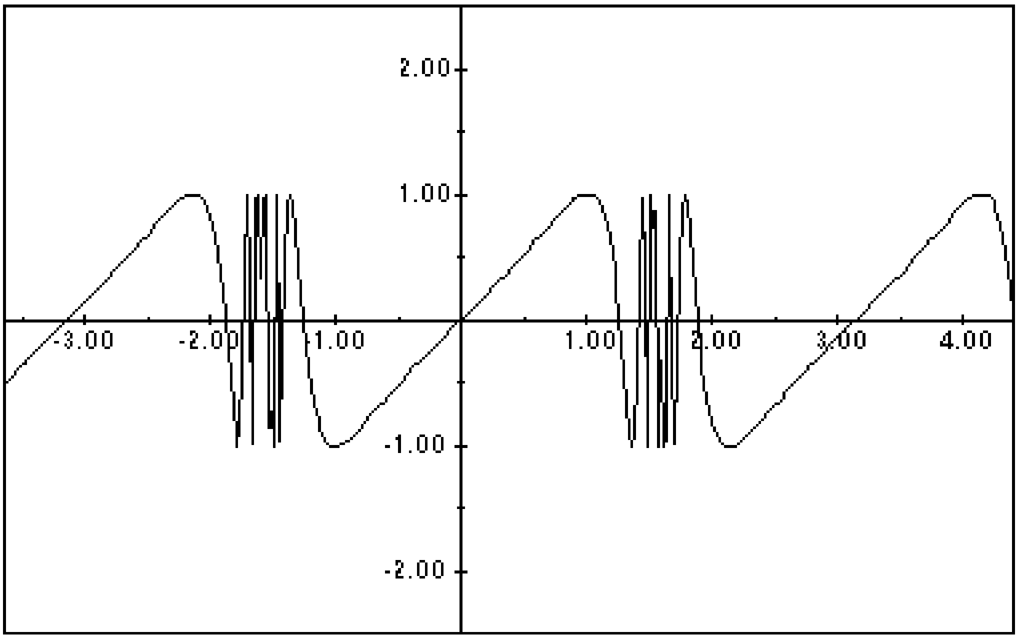


Figure 8.8: What is happening at the "fuzzy bits"?



When asked to describe "what is happening at the fuzzy bits" in the function **y = sin(tan(x))** (Figure 8.8), B4 used both *xFunctions* and the *HyperCard* graph plotter to examine the function, and then responded:

> CHAOS!!! is happening? As the sin and tan functions interact with each other. The function is periodic with a period which is large - wait and I will just find out how large...

She then returned to the graph plotter and used the "Trace" function to identify the period of the graph as 20 units. Like the function displayed in Figure 8.7, this problem required more than simply a visual use of the graph plotter to solve.

Even strategic use of the software tools, of course, does not guarantee a correct result. When B6 attempted a solution to the problem which Stephen had avoided, she used the graph plotter thoughtfully several times, and commented:

> Depending on how you square each equation you can get a different result. If you take x = 2 you can square it two ways: x^2 = 4 or x^2 - 4 = 0, for x - 2 = 0, the two ways are: x^2 - 4 = 0 or x^2 - 4x + 4 = 0 ... If we take the square of x = 2 to be x^2 - 4 = 0 and the square of x - 2 = 0 to be x^2 - 4x + 4 = 0 we find that the first equation has 2 solutions while the second equation has one solution.

Proceeding to the second part of the problem, she observes after graphing:

> If we square this equation we get x^2 - (6 - x)^2 = 0. This becomes x^2 - (36 - 12x - x^2) = x^2 - 36 + 12x - x^2 = 12x - 36 hence x = 3.

It seems likely that use of a computer algebra tool would have helped B6 avoid the frequent algebraic error, that $(a - b)^2 = a^2 - b^2$. It is also likely that her arrival at the correct solution using incorrect means was directly attributable to her use of the graph plotter, which allowed her to identify x = 3 as the required solution.

Whether complete or incomplete, correct or incorrect, the Group B preservice teachers engaged meaningfully and persistently with the open-ended mathematical explorations with which they were confronted. While the requirements of the assessment task undoubtedly contributed significantly to this interaction with both mathematics and tools, it does not entirely explain elements of perseverance and enquiry which appeared consistently. Rather it seems likely that **interest** and **curiosity** featured in the process.

Both groups of preservice teachers had chosen to be teachers of mathematics; they had recently completed extensive study of the subject and must be assumed to be at least competent in this regard and, presumably, interested in mathematics itself. Both groups interacted with essentially the same materials with basically the same available tools, and yet one group engaged meaningfully in open-ended mathematical exploration, while the other deliberately held back from such engagement. For Group A, it appears that "thinking like mathematics teachers" dominated their interactions with the tools and led to a failure to allow elements of curiosity and interest to surface. It is possible that, at least as perceived by one group of preservice teachers, mathematics is not something that mathematics teachers *do*, but simply something that they *teach*. This is consistent with a view of mathematics as a relatively fixed corpus of knowledge and skills which teachers, having acquired, are charged to pass on to their students, but not, it seems, to question or even to extend themselves.

My own use of the software tools, finally, involved exploring several of the same problems already described. I had attempted to select tasks appropriate and amenable to solution using computer tools by identifying activities which were unlikely to lie beyond the zone of proximal development for the intended users and yet which were not easily approached using traditional means. The problem displayed in Figure 8.7 was typical of such a task. It was not easily solved, either by traditional manipulative approaches or by simple viewing of a graph. At the same time, the solution was accessible, both using computer tools (in this case, table of values and the *MathPalette Solver* both produced immediate results) and by algebraic means, if the user approached the problem thoughtfully. In this case, the equation

$$\left(x^2-5x+5\right)^{x^2-9x+20}=1$$

could be solved by observing that, for any expression of the form $\mathbf{a^b}$ to equal 1, either a = 1 or b = 0. This readily leads to the identification by quadratic methods of solutions of x = 1, 4, 4 and 5. This problem was drawn from the 1988 NCTM Yearbook, *The Ideas of Algebra*, in which these are listed as the solutions. I had accepted this result until I happened to use the table of values and found that x = 2 and x = 3 also appeared to be solutions. The *Solver* which I had developed to complement the *MathPalette* also produced solutions at values of 1, 2, 3, 4 and 5. As deduced by B3, the additional solutions arise when it is realised that, for $\mathbf{a^b}$, if b is even, then a = -1 will also produce a solution. This occurs only when x = 2 or 3.

This provides an example, then, of a question which is most appropriate to examine using software tools. The solution is not accessible only through extensive computational capabilities on the part of the computer, but also by traditional means. In fact, the computer directs the user back to such means to understand the result, and so leads to increased understanding and a useful learning experience.

I continued to investigate this function, using the range of tools available, in case any other features of interest emerged. Use of the “trace” function of the graph plotter, the “substitute” function and the table of values suggested that further investigation may be warranted around the value of x = 1.36.

Function: $(x^2-5x+5)\text{^}(x^2-9x+20)-1$
12:07:14 PM
Button : ZoomIn
Trace graph with optionkey - observe values around 1.36.
CTRL-2 : 12:11:10 PM
Substitute: ...
CTRL-2 : 12:11:22 PM
Substitute: $(x^2-5x+5)\text{^}(x^2-9x+20)-1$ ...
If $(x^2-5x+5)\text{^}(x^2-9x+20)-1$ ...x = 1.37 ... then the result is -1
CTRL-2 : 12:11:57 PM
Substitute: $(x^2-5x+5)\text{^}(x^2-9x+20)-1$ ...
If $(x^2-5x+5)\text{^}(x^2-9x+20)-1$ ...x = 1.36 ... then the result is -1
CTRL-2 : 12:12:27 PM
Substitute: $(x^2-5x+5)\text{^}(x^2-9x+20)-1$ ...
If $(x^2-5x+5)\text{^}(x^2-9x+20)-1$ ...x = 1.371 ... then the result is -1
CTRL-2 : 12:12:58 PM
Substitute: $(x^2-5x+5)\text{^}(x^2-9x+20)-1$ ...
If $(x^2-5x+5)\text{^}(x^2-9x+20)-1$ ...x = 1 ... then the result is 0
CTRL-2 : 12:13:24 PM
Substitute: $(x^2-5*x+5)\text{^}(x^2-9*x+20)-1$ ...
If $(x^2-5*x+5)\text{^}(x^2-9*x+20)-1$ ...x = 1.38 ... then the result is -1

Table: 12:14:53 PM
y=$(x^2-5x+5)\text{^}(x^2-9x+20)-1$: 12:14:54 PM
Table : : 12:14:59 PM
Step Size : 0.01
function2 : y=$(x^2-5x+5)\text{^}(x^2-9x+20)-1$
Step Size : 0.001
function2 : y=$(x^2-5x+5)\text{^}(x^2-9x+20)-1$
Initial Value : 1.37
function2 : $(x^2-5x+5)\text{^}(x^2-9x+20)-1$
* SMA ( ) Exploring Algebra session : 12:19 PM, Fri, 28 Oct 1994
----
* mathpalette 28/10/94 12:19:53 PM
Algebra Tool 12:20:08 PM
Theorist Student Edition : y=$(x^2-5x+5)\text{^}(x^2-9x+20)-1$: 12:20:08 PM
****
Comment: 12:24:18 PM
Well - another interesting one. y=$(x^2-5x+5)\text{^}(x^2-9x+20)-1$ has solutions at x = 1, 2, 3, 4, and 5, but the graph suggests a zero around 1.36 (found using the trace function) just as it becomes undefined. The table of values produces a value of -1 for all values from 1.36- 1.37.
****

Questions such as this are extremely rich, mathematically, supporting and encouraging exploration by both teachers and students. Such questions demonstrate the potential for the use of computer tools to lower the barriers between teacher and students, offering the basis for them to become "co-learners" in mathematical exploration. While traditional questions associated with algebra learning tend to be closed, sequential and well-defined (and so discouraging the effective use of computer tools), questions such as those described here suggest that a critical aspect of "learning to use these new tools" will be in "learning to ask new questions". From the evidence of this study, strategic software use appears most likely to occur within the context of questions which are open-ended, rich in mathematical potential and yet accessible using both traditional and technological means. Within this study, such questions have included the high level problems mentioned above (most suitable for senior secondary and tertiary students), but may also be found among tasks available to younger students. For example, Ben engaged in a useful investigation - at the prompting of his tutor - as to whether $\mathbf{2x^2 - 4x + 2}$ should be classified as a "perfect square". In cooperation with his tutor, several different "definitions" of "perfect square" were recognised, which included:

- A numerical definition: a number which can be expressed in terms of two equal integral factors;
- A graphical definition: a quadratic function whose graph touches the x axis only once;
- A "factor" definition: a quadratic function which can be expressed using two equal factors;
- An "equation" definition: A quadratic equation with a single root (or two equal roots, depending upon your point of view).

As a result of this investigation, Ben came to realise that much of mathematics is subject to definition, and so whether a function may be considered a perfect square or not depends upon your point of view. He also gained considerably in his cross-representational facility, as he used graph plotter, number tools and computer algebra to explore the various representations associated with perfect squares.

At all levels, open-ended questions which are sufficiently challenging to require support and yet accessible enough to appear possible serve to encourage the type of software use which we describe here as strategic.

Confidence and Uncertainty

Finding the balance between tasks which are too difficult and those which are too easy has always been a hallmark of good teaching. Students invariably find the learning of algebra to be a challenging experience, its very symbolic nature placing demands upon formal thinking processes which are, for many high school students, still developing. It is not uncommon for students to spend a significant part of their time engaged in algebra learning in various stages of **uncertainty**. As has already been observed, students at all levels within this study were observed to be most comfortable with readily recognisable algebraic forms, especially equations which carried with them a signal to act in a specific predetermined way. Even such common forms as algebraic expressions (such as **4 - 3x**) proved a source of some uncertainty as students and preservice teachers were denied access to the action strategies available to them for equations.

When using computer software tools, these areas of uncertainty are likely to increase rather than decrease. In the majority of cases, the user is presented with a blank screen with little or no direction as to how to commence. In any given task, a student must first interpret the requirements of the task mathematically, deciding upon an appropriate course of mathematical action, but must additionally interpret the task technologically, choosing first between available tools and then, frequently, from a range of actions available within each tool. As the functionality of the tool increases, so does the range of potential choices and so, accordingly, does the level of uncertainty rise.

Particular software tools have attempted to address this problem. *Calculus T/L II*, for example, offers a unique support facility: having entered a function, equation, expression, data list or any of a number of possible objects, the program offers the user visible access to those actions appropriate to that object. Thus, for a function, for example, buttons for substitution, graphing, simplification, completing the square, differentiation, integration, limits and other mathematical actions are presented. Such an approach must serve to reduce the level of uncertainty experienced by the user, especially if that user is not mathematically sophisticated.

The interface offered by *MathMaster* proved advantageous for Patrick in developing his “Think of a Number” game. He first attempted to use *Theorist* to enter a series of mathematical operations which would eventually return him to the value of the original variable. The result was an expression of the form $\frac{2n - 6}{2} + 3$, which he found daunting. Using *MathMaster* Patrick was able to select each operation in turn, building up the expression by degrees, and having the program simplify it as he progressed, while keeping a visible record of each step in the process.

Within the instructional modules of the *Exploring Algebra* program and within the *MathPalette*, this problem was addressed directly through the inclusion of a **ToolKit** menu, which listed those mathematical functions available, from *Simplify*, *Expand* and *Solve* to *Animate, Areas under Curves* and *3D Graphs*. Selecting an option presents the user with a brief description of the available options and access to the tools which provide these. Selecting *Expand*, for example, the user is informed that expressions may be expanded using *MathMaster* or *CoCoA*, with an outline of how to access these features within these programs. Buttons are available for each which automatically open the selected tool, and the user may then carry out the desired operation. In this way, students are assisted in access to both the range of available mathematical actions and the available software tools which offer these.

The nature of the algebraic learning environment may be examined using constructs developed by Valsiner (1984) derived from Vygotsky's Zone of Proximal Development. Valsiner suggests that learning may be facilitated by first focusing the attention of the learner upon that to be learned by restricting the "Zone of Free Movement" (ZFM) and then encouraging desired actions which occur within a "Zone of Promoted Action" (ZPA). These elements feature strongly within the design of the computer-based instructional modules developed for this study. The computer environment serves as an inhibitory mechanism which actually promotes and encourages the use of the software tools as means of achieving desired mathematical actions. By making such tools familiar and easily accessible, students are encouraged to use them.

At the same time, the role of the tutor is a critical one. Wood's hierarchy of "levels of control" (Wood, 1986) suggests that the learner may be assisted across the zone of proximal development by judicious use of scaffolding by the tutor. Too much support and the learner fails to achieve independence; too little and the learner is overcome by uncertainty and becomes frustrated. Wood suggests that the support of the tutor be made *contingent* upon the response of the learner: each correct response by the student involves some withdrawal of tutor support, while each incorrect response initiates increased intervention. Wood defines five levels of control in this regard, from minimal intervention (involving broad general verbal suggestions) to actual demonstration by the tutor:

Level 0: No intervention

Level 1: General verbal prompt (“What else could you do?”)

Level 2: Specific verbal prompt (“You might use your tools here.”)

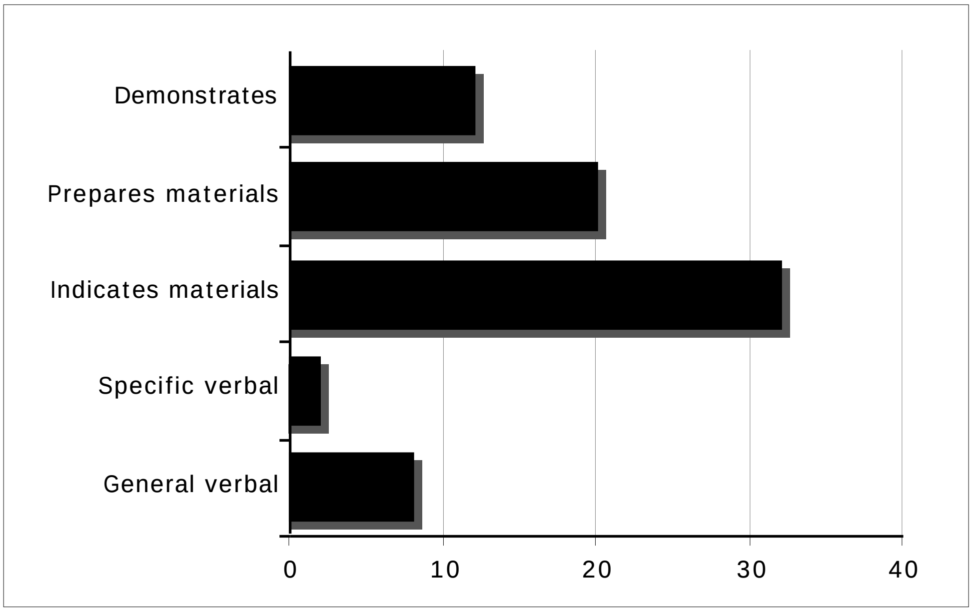
Level 3: Indicates materials to be used (“Why not use a graph plotter?”)

Level 4: Prepares materials (selects and sets up tool for student.)

Level 5: Demonstrates use of tool.

This model was adopted by the tutor in his interactions with the students in this study. Although intervention was intended to be minimal, Figure 8.9 displays a clear tendency towards dominant control by the tutor. Note that examples of “non-intervention” are not included in this display - those incidents in which the students took the initiative in selecting and using the tools. Such occurrences were, however, relatively rare and the principal conclusion to be drawn from the results in Figure 8.9 centre upon the general reluctance by the students to make free use of the available tools.

Figure 8.9: Frequency and degree of tutor intervention



If **uncertainty** is perceived as occupying one extreme in the mathematical encounters of individuals, then **confidence** lies clearly at the other pole. Under what circumstances are students likely to be most confident in their dealings with mathematics, and what is the relationship of tool use with student confidence, particularly with regard to their own solutions? Increasingly the attention of the researcher was drawn to issues associated with student confidence after several participants commented upon their own perceptions of whether a particular response was or was not correct. Specific data were gathered which proved revealing of the relationship between confidence and tool use in the learning of algebra.

At various times each of the students engaged in “quizzes” or “reviews” as part of their progress through the instructional modules. These generally consisted of ten multiple-choice questions which simulated traditional assessment components which might be associated with particular topics. In order to further simulate “school-type” assessment, in most cases the software tools were not immediately available. Students were asked to attempt the questions themselves, unaided. They would then either select what they saw as the correct response or, in some cases, check their answer with selected tools before selecting from the given responses. As has already been mentioned, attempts were made to motivate the students to try to attain the highest possible scores: two marks, for example, would be awarded if their first response was correct, and a mark deducted for each incorrect response. It was sought to encourage the students to verify their answers (using the software tools if necessary) rather than simply offer their first response. Such validation appears compatible with a high level of responsibility for their own learning, and conducive to active participation in the learning process.

Consider the following excerpt from Andrea's record, in which she is attempting two questions from the "Elementary Algebra Review":

> Comment: 5:03:35 PM
> Expanding (x-2)(x^2+2x+4) I got x^3-8 first and then changed it to +8, then back to -8 (after seeing the answers). I am about 45% confident.
> ************************
> Comment: 5:07:02 PM
> After checking it on the computer I am now 85% confident.
> ************************
> Comment: 5:10:56 PM
> For (6x^4yz^2)/(-12xy^2z) I got (x^3z)/(-2y) and am 85 % confident.
> ************************

Andrea was permitted to use the computer tools before nominating an answer. Her confidence after checking these two questions rises, in the first case from 45% to 85%. Note, however, that even after checking, she is not 100% confident in her response. This pattern continued through the subsequent "Elementary Trigonometry Review", in which most (validated) responses again rated only 85% (and one 90%)

confidence, while those results which were not validated rated only 40-50% confidence. Clearly, Andrea is generally not confident of her answers to such questions. The fact that, even after validating answers using appropriate computer tools, she still does not express 100% confidence appears to reflect this general lack of confidence in her own abilities, rather than her perception of the computer tools as being inadequate.

The same pattern continued, even when attempting the "Beginning Algebra Review", which involved questions which, while not difficult algebraically, appeared in a format which Andrea found unfamiliar. Later in this review, however, she appeared to become more comfortable:

> I think it is y = 2x+3: 90%
> ************************
> Button : E*
> Card: BA quiz 9
> ************************
> Comment: 5:11:28 PM
> y=3x-2: 3x-2=43 -> x = 15: Confidence 85%
> ************************
> Table: 5:12:22 PM
> function2 : y=3x-2
> ************************
> Comment: 5:13:12 PM
> Confidence 100%: The computer proved to me that it is right.
> ************************
> Button : B*
> Card: BA quiz 10
> ************************
> Comment: 5:14:26 PM
> 100% because it is just like simplifying.
> ************************
> Button : A*
> Card: Score Card
> Button : Back to the Start?
> Andrea 25/7/94 5:15:20 PMThe score was 15
> We used it to check my answer and to increase my confidence rating. It was effective, since after I plugged in the formula which I thought it should have been and I saw that the values were correct.

Her comment that the “computer proved to me that it is right” suggests, not only faith in the technology, but confidence in her own ability to use the tool (in this case, the table of values) appropriately. As has been observed elsewhere, the table of values proves a most convincing representation.

Finally, Andrea’s attempt to answer questions from the “Curves Review” with the assistance of the graph plotter saw her increase her confidence rating from 40% to 100% in one case, because “the computer proved it and I'm not going to argue with the computer”.

The pattern of tool use was repeated consistently, not only for Andrea, but for each of the student participants: use of appropriate computer tools resulted in an increase in confidence in the proposed answer. For Ben , for example,

> For 8p^3+64 my CONFIDENCE is 100% because I did on the computer...
> My CONFIDENCE in solving the equation went from 80% to 100%...
> 1993: 2U Q1(a) 8.1 CONFIDENCE 50%, checked twice - confidence 100%

Stephen, too, while preferring to trust his own manipulative skill, nonetheless expressed an increase in confidence after using *Theorist*, on one occasion from 40% to 90-100%.

The evidence clearly supports the assertion that use of appropriate mathematical software resulted in increased confidence in the solutions presented by the students, even to quite complicated questions. A disturbing aspect of this data concerns the surprising frequency with which students expressed levels of confidence at or below 50% with regard to their answers to questions reviewing material with which they were familiar. Coupled with this lack of confidence in their own answers is the persistent failure of these students to seek to verify their results unless prompted to do so. Andrea proved the exception to this rule: she regularly and spontaneously availed herself of software support when she was uncertain of a result. For Stephen and Ben, however, their reluctance to make use of computer assistance, even when they expressed little confidence in their results, suggests again the influence of a culture of mathematics learning which devalues the use of external tools (the calculator appears to be the exception here) and within which the primary goal appears to be to produce an answer, whether correct or not. Students such as Ben in particular appear to view the responsibility for their own mathematics learning as residing with someone other than themselves.

An Overview of Tool Use

It is now possible to view the incidents of tool use which occurred across all participants in terms of level descriptors which arise from the data. As previously described, specific examples of software use may be perceived as occupying positions at various points along a continuum, which may be described in the following terms:

Level 0: *Non-Use:* Although the software is available and appropriate, and the user has sufficient skill to use it, no use is made.

Level 1: *Passive*: The user is content for the tools to be operated by another, but takes no personal initiative.

Level 2: *Random*: Use is not goal-directed and bears no relation to the context.

Level 3: *Reflexive*: The user makes superficial and automatic use of appropriate tools.

Level 4: *Strategic*: Use of the tools is deliberate, goal-directed and insightful.

The relative frequency of occurrence of Levels 1 to 4 for the participants is displayed in Figure 8.10. The number of actual incidents associated with each level is considered as a percentage of the total number of incidents of tool use by each individual. Thus, of my own seventeen incidents of tools use, 4 were classed as reflexive (23.5%) and 13 as strategic (76.5%) .

The students were alone in displaying passive use, and showed no occurrence of random use - both owing to the presence of the tutor in every interaction with the tools. Incidents of strategic use by the students were more frequent than those of reflexive use (50% as opposed to 34%), while this situation was reversed for the preservice teachers, for whom the most common interactional type was reflexive use. As expected, Group B showed a greater tendency towards strategic use of the tools than did Group A, (23% of interactions as opposed to 13% for Group A).

Figure 8.10: Tool Use as Percentage of Incidents

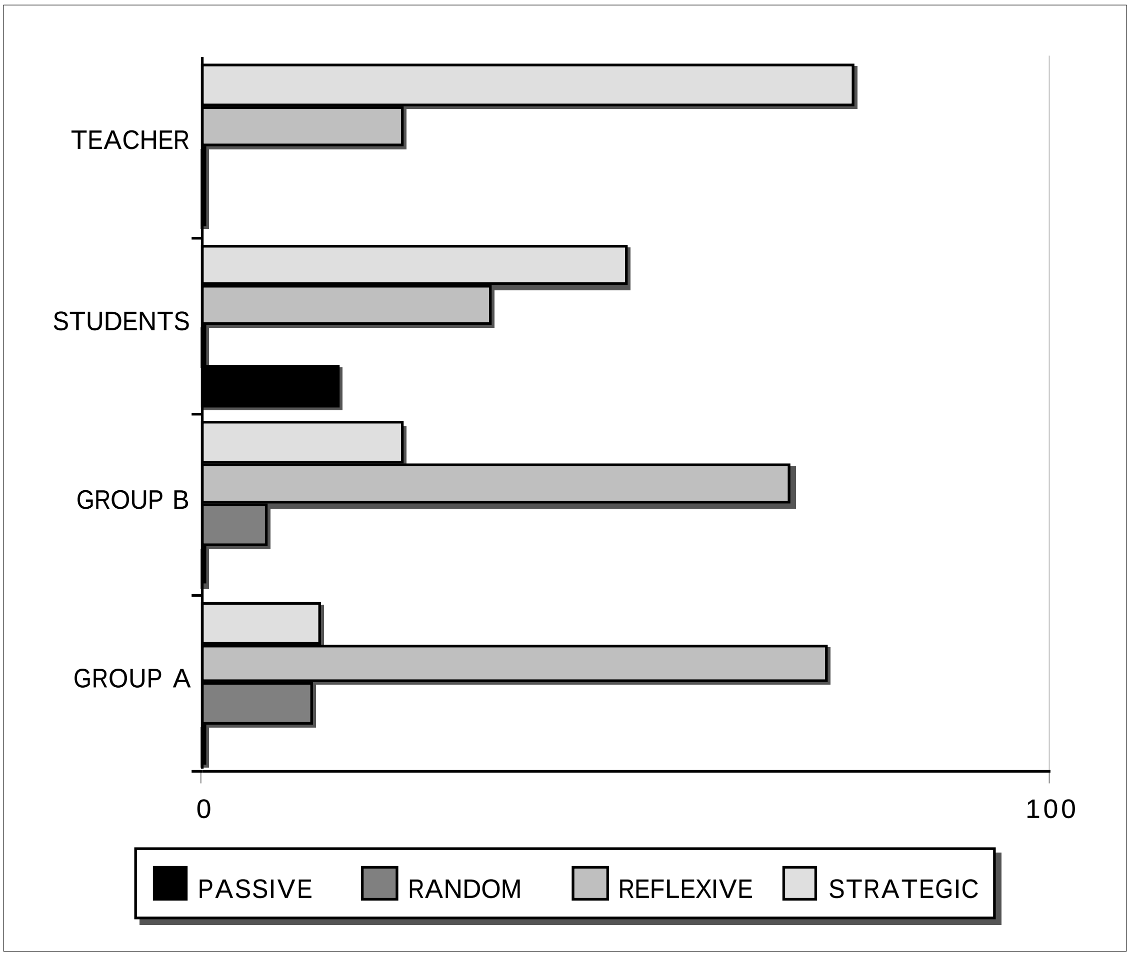


Figure 8.11: Patterns of Tool Use (Group A) - Number of incidents

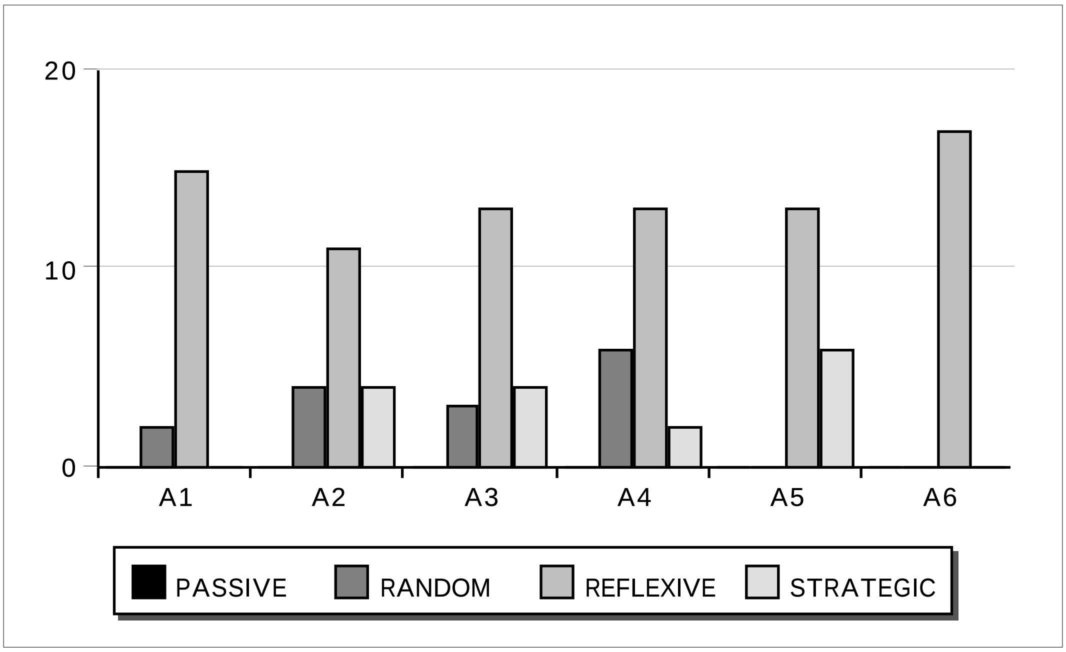
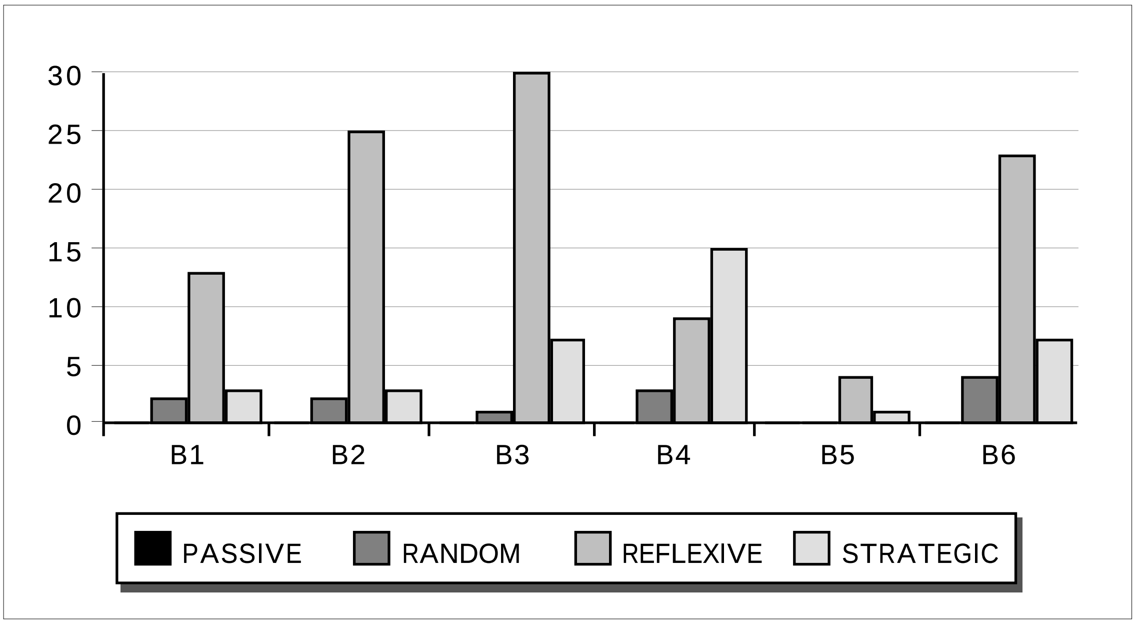
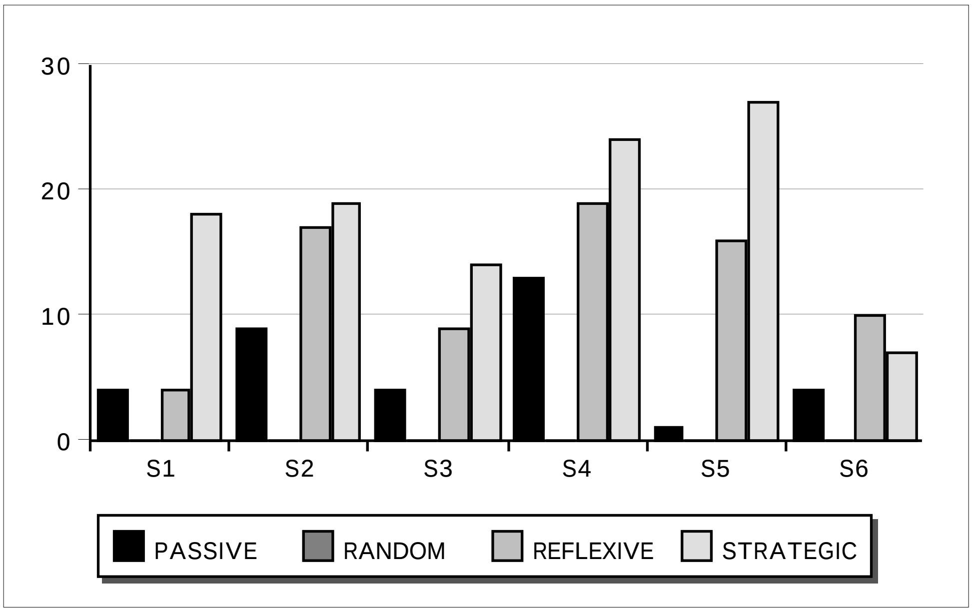


Figure 8.12: Patterns of Tool Use (Group B) - Number of incidents



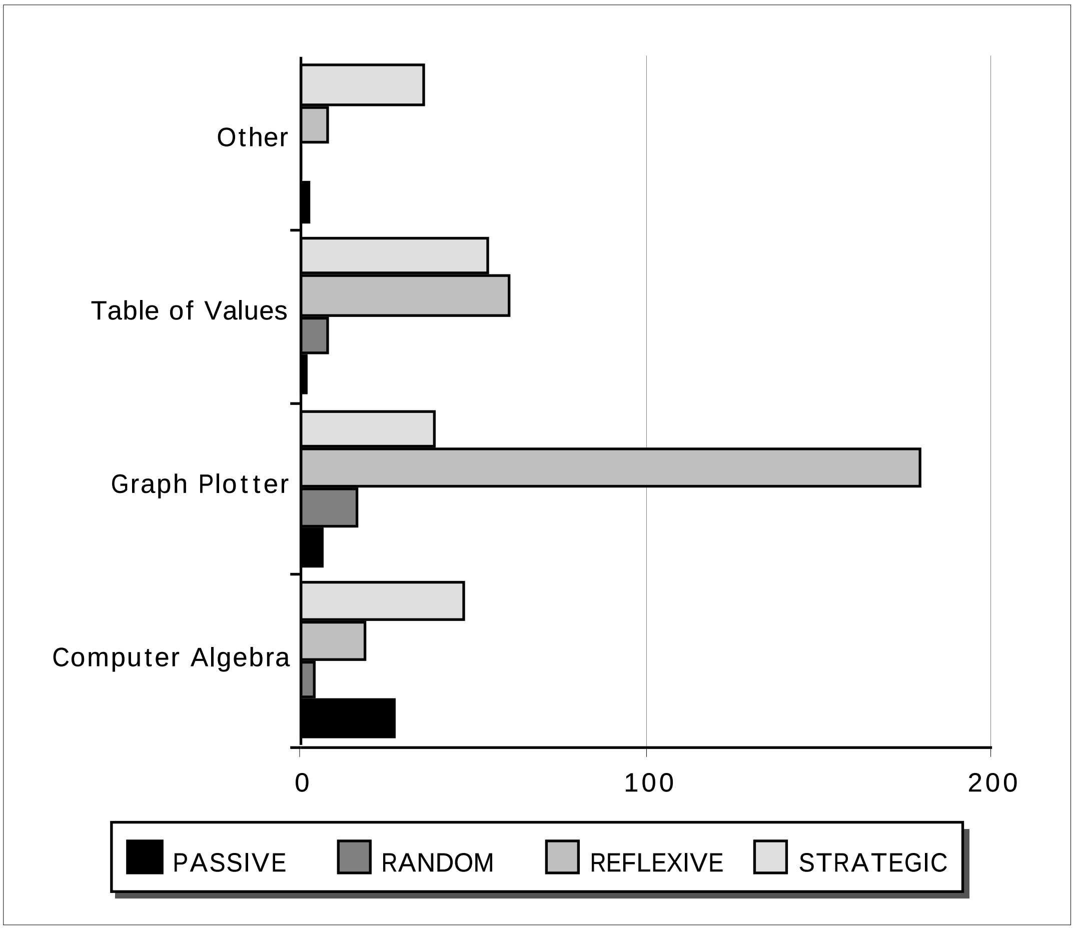
All Group B participants showed some strategic use of tools, while both A1 and A6 showed no evidence of such use. Tool use by both groups was dominated by reflexive activity, with some occurrence of random use in their early encounters.

Figure 8.13: Patterns of Tool Use (Students) - Number of incidents



The pattern of tool use for the students was quite different from that of the preservice teachers. While all engaged in some passive use of the tools (as the tutor demonstrated them), strategic use dominated the interactions for all but Patrick (S6). Andrea showed least reflexive tool use and Tony the highest incidence of strategic use (largely through his interactions with the utilities available within the *Exploring Chaos* module.) It is likely that the influence of the tutor must be considered as a significant factor in influencing the students in their high level use of the available tools; such was a specific and explicit priority of the instructional component of the study. The reluctance of most students to freely engage in tool use has already been noted; the fact that, in spite of this hesitancy, all students engaged meaningfully in mathematical interactions with the software must be considered a sign of success with regard to the technology-rich algebra learning environment created.

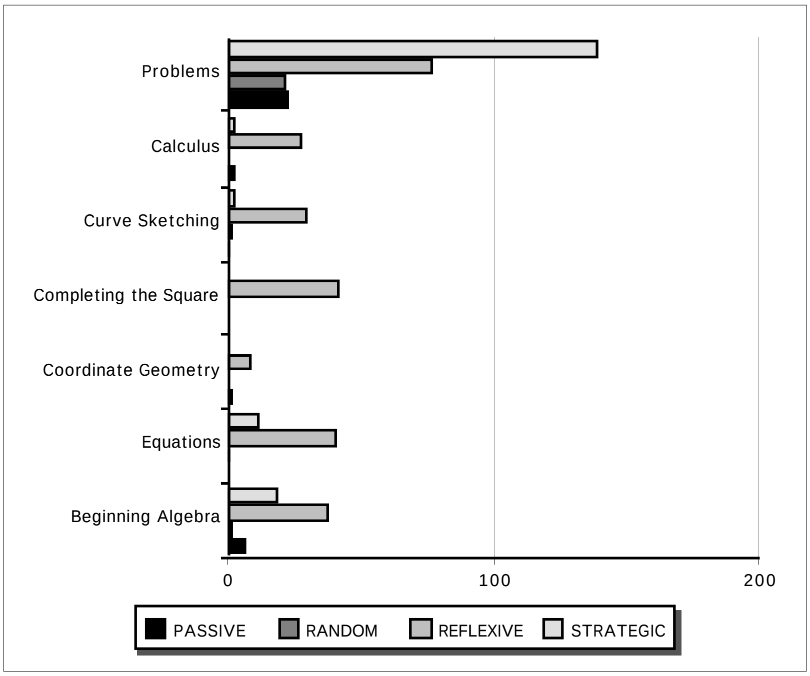
Figure 8.14: Breakdown of Tool Use by Tool Type



When the levels of tool use are considered in relation to the major tool types used for this study (Figure 8.14), several features appear significant. The high incidence of passive tool use for computer algebra reflects the priority accorded this tool type within the study. It suggests, too, that the other available tools were seen as requiring less specific instruction. The high frequency of reflexive use related to the graph plotter supports the observation that most use of this representation was of a visual rather than an analytical nature - users would quickly observe and draw required information from the graphical image and then move on to the next task. It is possible, too, that automating access to the graph plotter using *hypertext* facilities within the modules may have served to encourage reflexive tool use at the expense of more considered and analytical approaches.

Consideration of level of tool use in relation to content area adds further support to this criticism of the ease of access to the graphical representation. Those modules which most encouraged superficial viewing of graphs by simply "clicking" on algebraic forms as they occurred (*Curve Sketching, Completing the Square, Coordinate Geometry* and *Calculus*) were those most strongly associated with reflexive use of the tools. As has already been observed, strategic use appears to be most commonly associated with open-ended tasks which offer opportunities for **exploration**, while reflexive use most often coincides with those activities involving highly structured and predetermined **instructional** sequences.

Figure 8.15: Breakdown of Tool Use by Content Area



Nine
Toward a Grounded Theory

The many pieces of the puzzle have now been laid out and examined. Their properties have been teased out, leaving only the final stage of the process - putting the pieces together in relation to each other in order to form a coherent whole. To this end, the grounded theory method offers what is termed the "paradigm model" as a guide to those features considered most important in developing a theory which is to be dense and coherent, offering an integrated and explanatory description of the phenomenon under consideration. Such a theory has been lacking up to this point, with concerns related to classroom use of technological tools taking precedence over the necessary study of individuals interacting with mathematical software.

The action research problem which gave rise to this study involved "learning to use new tools". The grounded theory of software use proposed offers substantive contribution in this regard. For practitioners seeking to use such tools to enhance their own teaching and learning of algebra within technology-rich environments, the detailed case study descriptions and the subsequent grounded theory allow them to experience vicariously the interactions and encounters, the successes and failures of the teacher as researcher in this context. They may then judge for themselves the extent to which these experiences are congruent with and informative of their own situations. The study offers, too, a wealth of detail regarding a new and rich aspect of mathematical pedagogy. It suggests new questions and new implications for further research in an increasingly significant domain.

The paradigm model offered as the principal tool for grounded theory analysis (Strauss and Corbin, 1990, p. 99) requires detailed consideration of the phenomenon in question in relation to causal, contextual and intervening conditions, action/interaction strategies and, finally, consequences. As outlined in Chapter One, this model relates the various components to offer a unified and integrated whole.

(a) PHENOMENON or CORE CATEGORY

This chapter first considers the core category, or “phenomenon” for this study: mathematical software use. As a result of the analysis of data from the various respondents, it is now possible to offer a detailed description of the nature of such use, in which various contributing factors define its frequency and form. This form is recognised as composed of quite distinct dimensions, ranging from *non-use* and *passive use* at one extreme, to *strategic use* at the other. As previously noted, this phenomenon of *strategic software use* occupies a position of central concern in the present study, representing as it does a powerful and desirable condition for learning.

(b) CAUSAL CONDITIONS :

Having defined the central phenomenon in terms of its specific dimensions, it may now be situated in relation to the key *causal condition* which defines its occurrence, a cyclic framework which relates the mathematical situation, its interpretation and subsequent action on the part of the learner, and the subsequent evaluation of the result of this action. This places the tool use in relation to both user and learning environment.

(c) CONTEXT

The nature of the tools themselves and of the learning environment provide the *contextual conditions* under which the phenomenon takes its specific form. It is possible now to identify those features associated with "good" mathematical software in the context of the algebra learning experiences encountered in this study. It is possible, too, to identify desirable features of the learning environment, under which conditions, *strategic software use* is considered most likely to occur.

(d) INTERVENING CONDITIONS

Aspects of mathematical and pedagogical thinking served as the key *intervening conditions* identified in this study. Of the former, preferred imagery and the extent to which various algebraic forms signalled action strategies on the parts of the users were most significant; beliefs concerning the nature of mathematics and algebra, and the ways in which these are best learned were also identified as critical in determining the extent and form of mathematical software use.

(e) ACTION/INTERACTION STRATEGIES

The ways in which the various individuals and groups actually *used* the available software tools was considered in detail in Chapter Eight, and these are here identified as *action/interaction* *strategies*. It is difficult to extrapolate beyond the confines of the present sample, but quite distinct usage patterns were identified, and these may certainly inform the actions and planning of other practitioners, and perhaps form a basis for subsequent research. It *is* possible to identify two ways in which the available tools (especially computer algebra tools) were found to be most effective in this study: as support for extended mathematical manipulative processes (such as equation solving and completing the square), and as support for investigation and exploration of problems and mathematical concepts, freeing the user of manipulative constraints.

(f) CONSEQUENCES

Specific positive and negative consequences of the use of the tools in the current context may be clearly identified. Positive results included increased confidence and improved representational repertoires on the part of all participants. At the same time, some evidence was found of misunderstandings and over-dependence on the tools by some of the participants. Consequences must be viewed within the framework of the various contextual conditions already identified.

The network of relations thus created ensures that the subsequent theory is dense in both descriptive and explanatory power, raising the level of abstraction from initial grounding in the data to a well-developed substantive theoretical position. The resultant theory is then considered in the light of related research, and implications for practice and further enquiry.

The Phenomenon of Mathematical Software Use

Within a given algebra learning context, software use is most likely to take the form dictated by a particular tool type. In the present study, these were primarily:

- Algebra tools (principally for representation and manipulation)
- Graph tools (for representation)
- Number tools (for representation)
- Utility functions (particularly facilities for substituting, solving and calculus available within the *MathPalette* and versatile tools such as *xFunctions*.)

Within these various tool forms, a range of properties has been discerned as defining the nature of the tool use. These were found to include:

- **purpose** (whether for verification of results, for representation, manipulative support, exploration or simply for convenience);
- **goal-directedness** (the extent to which goals were well-defined and achievable, and the persistence shown in working towards these);
- **versatility** (particularly with regard to the use of a range of tools and access to several appropriate representations);
- **confidence** (in both use of the software tools and in the mathematical results obtained);
- **motivation** (both *intrinsic*, resulting from interest and curiosity, or *extrinsic*, resulting from the demands of teacher or assessment).

The specific dimensions, or "levels", of mathematical software use have already been described in the context of their occurrences within the data. It is now possible to define these dimensions in terms of the properties of tool use given above.

Strategic software use may serve a variety of *purposes*, involving at different times all of the categories mentioned above. While open-ended exploration is most readily associated with this level of software use, it also frequently involves verification of results, which is active and often versatile (as the user deliberately and thoughtfully uses available tools as means to validate findings and to support conjecture). Strategic tool use also involves both representational and manipulative actions as mathematical responses.

Strategic use is most clearly defined by its highly goal-directed nature. The selection and use of available tools is deliberate and thoughtful, with clear **intention** to achieve a particular desired end. It is frequently **versatile** in the use of both varied representations and a range of appropriate mathematical and computer-based strategies. Verification of results is commonly achieved through multiple sources. Confidence associated with strategic use is high, both with regard to the mathematical strategies deliberately chosen and with regard to the results achieved, and motivation may be expected to be dominated by intrinsic factors, especially interest and curiosity. While such use may have been initiated from external sources (such as the prompting of a teacher or tutor, or the requirements of an assessment task), without this critical feature of intrinsic motivation, the tool use appears unlikely to exhibit the important element of **persistence**, which appeared as a significant factor within this study.

Reflexive tool use appears more limited than strategic use in most regards. In terms of purpose, reflexive use is most commonly associated with verification of results and representation, and least commonly with exploration and manipulation. While goal-directedness may be high in some instances, reflexive use commonly features a lack of persistence on the part of the user, and a limited representational repertoire. In fact, such use was observed most commonly associated with a single representational category - the graph plotter. Confidence varies with such use, from very high to very low, and motivation for such use may be expected to be external, with less personal commitment on the part of the user than was observed for strategic use.

Random use of mathematical software may be considered a sub-category of **reflexive** use. It was found only among the preservice teachers who, especially in their early encounters with the software tools, explored the limits of the "zone of free movement" offered them within the confines of the computer modules, and used the tools freely without regard for curricular context, or even any observable goal. Such use, while occasionally versatile, was observed to be low in goal-directedness and persistence.

Passive use was most clearly defined by being externally motivated. The extent to which other factors were demonstrated was dependent upon the intervention of the teacher/tutor, rather than the individual user. While confidence may have increased as a result of such use, it was also observed to result on occasion in decreased confidence and lack of understanding, particularly when the tool use extended beyond the zone of proximal development of the student. Such use by the tutor also served to discourage independence and initiative on the part of the students , and so led to limited personal commitment.

Non-use of software tools was difficult to examine directly in the context of this study since its occurrence can only be inferred rather than observed. Nonetheless, it is a most significant aspect in terms of understanding the phenomenon of mathematical software use, and must be considered at this point. The most specific instances of non-use of available tools were observed in association with the many review exercises undertaken by the students. Since they had been encouraged both to ensure to the best of their abilities that their responses were correct and to use available tools to assist in this regard, the frequent failure of individuals to do so when answering incorrectly may reasonably be considered as examples of this type of tool use.

Stephen, for example, encountered fifty review questions on topics ranging from beginning algebra and equations to general algebra reviews and the “stress test”. Of these, he answered fourteen incorrectly, but used available tools (computer algebra and table of values) only four times, and these when prompted specifically. Of Ben’s thirty-five review questions, seven were answered incorrectly and levels of confidence frequently dropped to 60 or 70% prior to selecting an answer, and yet these factors were not sufficient motivation for software tools to be used.

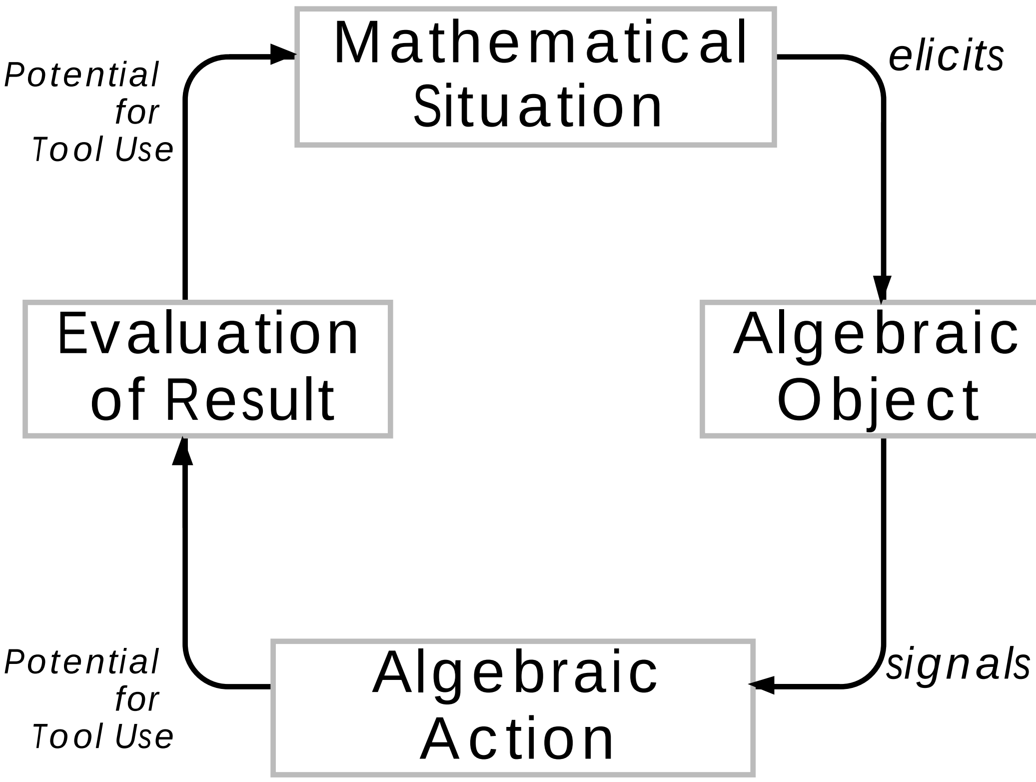
Andrea demonstrated that use of the computer tools did not guarantee a correct result every time - she showed no reluctance to use the available tools when she was uncertain of her result. Of her fifty review questions, she made only five errors since she regularly and on her own initiative verified her results using computer algebra, graph plotter or table of values, whichever she considered appropriate.

It is a primary concern of the theory of mathematical software use which follows to offer some insight into the conditions under which the various dimensions of tool use occurred, and to seek to explain those factors which may have served to both encourage and impede such use. Now that the nature and dimensions of the phenomenon have been detailed, it is appropriate to consider those conditions under which it may be observed.

Causal Conditions

Figure 9.1 describes a cyclic framework within which the potential for mathematical software use may be usefully situated. This framework is made up of four components: an appropriate **mathematical situation** in this context is considered to be one which elicits recognition of an **algebraic object** (most commonly an equation, expression, function, graph or table of values). Such an object may be explicit or implied. The former is commonly associated with an algebra learning environment dominated by an instructional perspective, composed of carefully sequenced and deliberate learning activities. Such an environment corresponds closely to the first two **stages of learning** as proposed by van Hiele - the stages of *information* and *guided orientation*.

Figure 9.1: A causal framework for mathematical software use



An implied algebraic object demands both recognition and interpretation on the part of the learner. While such high level cognitive activities may be found at the lower levels of van Hiele's stages, they are more likely to occur within contexts of *free orientation* and *integration*. Failure on the part of the learner to recognise an algebraic object within a particular mathematical situation may not mean that no further mathematical actions can be effected. It does, however, negate the possibility of software tool use within that context, since such use requires an object upon which to act.

Consider, for example, the "unemployment" problem from the module *Something to Think About* which was attempted by both Stephen and Ben. In this problem, information is presented regarding changes to unemployment rates in a hypothetical country over a period of weeks following the election of a new government. While the information is not amenable to use with available computer tools, careful interpretation leads to the recognition of features more commonly associated with curve sketching and calculus, but within an unfamiliar context. Once recognised, the activity invariably resulted in insights regarding, not only the applications of calculus to curve sketching, but also as to the nature and purposes of the important concept of the derivative as a rate of change. This was a rich mathematical exploration which did not require software use, but certainly involved mathematical actions and thinking.

Recognition of an algebraic object may be considered a condition which is necessary but not sufficient for the occurrence of mathematical software use. The object itself, then, must signal a **mathematical action** from the repertoire available to the individual learner and *the nature of the object as perceived by the user will influence the way in which it functions as a signal to act mathematically*. Such a repertoire will contain traditional algebraic actions (simplify, expand, factor, solve, substitute, sketch, differentiate or integrate). Within the technology-rich learning environment created for this study, however, all of these actions were also available using software tools, in addition to representational actions enhanced (and made possible) by the computer, especially graphing, tabulating, and even animating. The extent to which the individual learner has **integrated** both traditional and computer-based mathematical actions must be considered a critical feature in the use of software tools. The potential for tool use at this point is largely dependent upon the extent to which such integration has occurred. Of the students, only Andrea appeared to display such integration, choosing freely from both traditional and computer-based approaches to given mathematical situations. While the level of integration must clearly be influenced by the algebraic thinking of the individual, the results of this study suggest that factors associated with pedagogical thinking (attitudes and beliefs concerning algebra and algebra learning) were far more influential as determinants in the use of available tools.

Having recognised an algebraic object within a given mathematical situation, the learner then chooses from a range of available actions (which may or may not involve the use of software tools). Such action produces a result which must be **evaluated**, usually in terms of an expected outcome. *It was common at this point for students to use available software tools for purposes of verification of results which had been obtained by traditional means.* Potential for tool use at this point was high, as the use of the computer for purposes of verification of results appeared to be perceived generally as both helpful and legitimate, in contrast to the use of the software to support and replace traditional approaches.

Traditional mathematical actions as observed in this study tended to move relatively quickly to a point of closure. Algebra was commonly associated with obtaining an "answer", usually through the application of a well-defined sequence of steps. Such a perception is seen as largely incompatible with a focus within the learning environment upon open-ended exploration. In fact, the readiness with which even high ability students such as Stephen would conclude their computations, even while expressing less than full confidence in their results, was noted as a source of some concern. When limited to traditional methods, then, the stage of **evaluation** appears likely to conclude the mathematical process with brief verification using whatever means are available.

Use of computer tools, however, served to encourage a cyclic aspect in this process. Even when used only for verification of results through use of an alternative representation (such as viewing a graph to check the solution of an equation), the user is presented with what is effectively a new mathematical object or situation, requiring further interpretation and the possibility of subsequent action. If the user exhibits those characteristics associated above with strategic software use (goal-directedness, versatility, perseverance and, most importantly, curiosity) then the stage of evaluation may be expected to lead to a new sequence of mathematical action, interpretation and reflection, and such was observed frequently within the data.

Figure 9.2: Mathematical software use situated within a causal framework

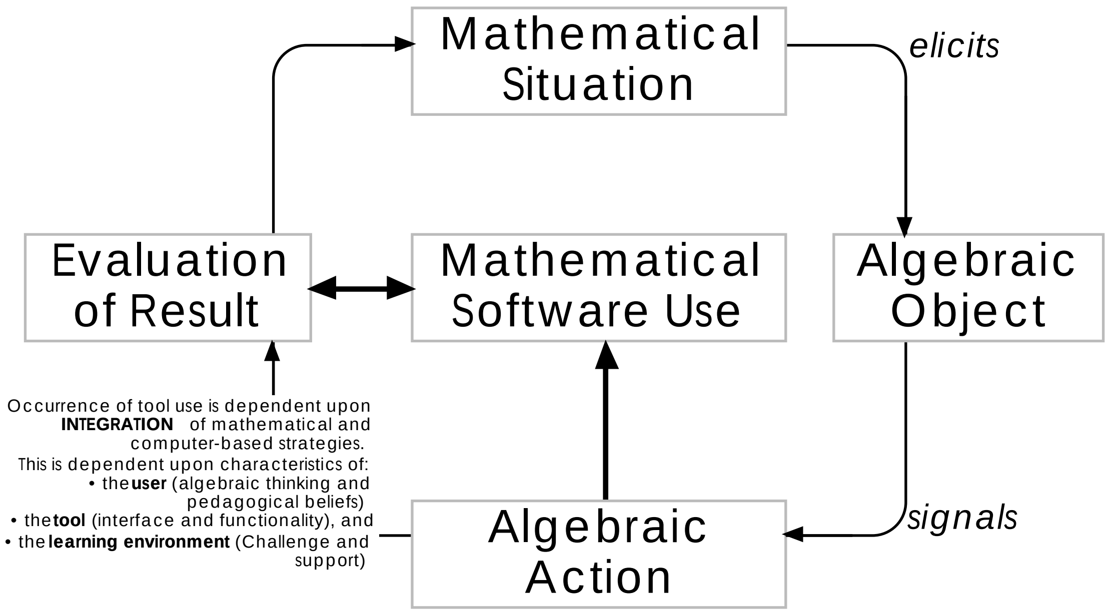


Figure 9.2 situates mathematical software use within the causal framework described. Both the stages of *action* and *evaluation* are likely to give rise to tool use, the latter in a potentially cyclic way. The potential for tool use, however, is limited by the degree of *integration* present on the part of the user, dependent upon characteristics of user (algebraic and pedagogical thinking), tool (interface and functionality) and learning environment (the balance between challenge and support).

No single factor, then, can guarantee that mathematical software use will occur. Rather, several key aspects of individual thinking play important roles in deciding this phenomenon, and these will be detailed as intervening conditions in this theory. Prior to this, however, it is necessary to examine the role of specific **contextual conditions** within this process.

Contextual Conditions

The context within which mathematical software use occurs is considered in this study to be dominated by those factors associated with both the mathematical software tools themselves, and the algebra learning environment within which they are available. Both factors served to define and direct the interactions of individuals with mathematical tools.

With regard to the software tools themselves, the twin features of **interface** and **functionality** appeared to figure strongly in determining their use by both students and preservice teachers. Simplified entry of algebra forms appeared to encourage such use - even the more experienced users commonly made errors of entry, typing 4x instead of

4*x when required. It was clear that the algebraic form required by the software should mimic as closely as possible the usual written form. Thus, 4 - 3x is preferable to 4 - 3*x, and x^2 - 4x + 4 is to be preferred to x^2 - 4*x + 4. All participants adopted the use of the option key for the placing of exponents quickly and easily, providing simple access to the two-dimensional formatting which is the norm for algebraic forms.

The creation of the "palette" as a means of simplified entry of algebraic forms was an important aspect of the development of the algebraic learning environment for this study. This method is used by such quality *Macintosh* applications as *Theorist* and *ANUGraph*, and appeared to offer an ideal means by which quite difficult algebraic forms could be entered without recourse to specialised code or instructions. The overall response to the palette, however, was disappointing. Although participants used it when prompted, it was generally found to be both cumbersome and slow, and keyboard entry (in simplified form) was invariably the preferred option. Particular problems were found in using the palette to create complicated expressions, especially those involving fractions and exponents. Although improvements were made in response to observed difficulties (such as automating the closing of parentheses), the palette as a form of algebraic entry would need to be far more intuitive than appears to be currently the case for it to be preferred over simple keyboard entry procedures.

The functionality of the various software tools is, in some respects, a complementary issue with interface. A significant factor in the strong preference shown by all participants for the graph plotter as a mathematical tool must be the fact that it does a single job well. Tools which offer a wide range of mathematical choices are likely to act to increase uncertainty in student users. As Goldenberg (1988b) observes regarding multiple representational software, "(w)hile potentially reducing ambiguity, multiple representation also presents a student with more places to look and is potentially complicating and distracting" (p. 136). The same may be applied to much available computer algebra software, frequently offering hundreds of potential choices for action. At the same time, a broad range of functionality is a useful feature, and so the critical factor appears to be the **accessibility** of the various features. The range of available functions should be clearly visible to the user and simple to access. Thus, programs such as *CoCoA* which require specific command-line instructions and offer the user a blank page and no useful menus from which to access commands must be considered a poor choice. Even the *Theorist* interface fails to support access to the full range of available functions in a way which is intuitive to students.

As noted previously, the interface of the program *Calculus T/L II* appears to satisfy this demand most effectively, making available those functions appropriate to the current algebraic object, and so actively reducing levels of complexity and uncertainty. It is unfortunate that this program requires entry of algebraic forms in unsimplified format.

The development of the *ToolKit* menu across all instructional modules was a deliberate attempt to reduce uncertainty and provide access to available mathematical functions. In this way, algebraic forms could be entered in simplified form (or using the palette if desired), and then pasted into other software tools which are linked through the *HyperCard* interface. While *xFunctions*, *Theorist*, *MathMaster* and other commonly used software tools may each require entry in a different form, students using *Exploring Algebra* were provided with a simple and consistent format for entry of algebraic forms which could then be pasted into any of the available tools.

The evidence of this study suggests certain features which must be considered desirable in algebraic software. The **interface** must be clear and intuitive, with available functions clearly visible and easily accessible. This is most readily achieved through the use of pull-down menus which list available mathematical actions (and ideally such actions may be accessed through visible on-screen buttons as well).

Entry of algebraic forms must be simple and closely approximate written forms. The addition of some intuitive method for entry of exponents (such as the option key or the “up arrow” key) is preferable to the use of computer characters, such as “^”. Display of the algebraic form must utilise full two-dimensional formatting, allowing users to verify that they have entered the desired expression correctly.

Both graphical and tabular representations must be open to manipulation, permitting adjustment of all parameters in addition to quick and easy facilities for “zooming in” and “zooming out”. Axes must be clearly labelled for graphs, and options should be available both for grid lines and labelling in multiples of π. Flexible entry of algebraic forms appears to be desirable as a means of encouraging versatile thinking regarding algebraic objects. Thus, while an equation such as “y = 2x - 1” may be the preferred form for both graphing and tabulating, it should be possible to enter alternative forms such as “2x - y - 1 = 0” and even expressions such as “2x - 1”.

Manipulation of algebraic forms should be under the control of the user, while supported by the software. If an equation is being acted upon, for example, the program should automatically act upon both sides, reinforcing and supporting traditional methods. A record should be visible of each step of the interaction, allowing students to follow the process by which their result was achieved. This computer-based support and display of each step of an algebraic process was considered by the students in this study to be the most useful feature of the computer algebra software which they used. At the same time, there were frequent occasions within the tutorial situations in which a quick result was desired in order to verify a computation. At such times, a computer-generated result encouraged both verification and further exploration. For this reason, *ToolKit* facilities were added for a range of common mathematical processes which involved quite complex computations, including equation solving, derivatives, areas under curves, and even loan repayments. Having ready access to such features permitted strategic use of the software as a convenient means of checking both results and conjectures.

It appears that "good" algebra software should support both the development of algebraic processes and convenient access to a variety of algebraic results. It should offer at least symbolic, graphical and tabular representations, and facilitate movement and transfer of information between these. Access to the various mathematical functions of the software must be clear and intuitive in order to minimise uncertainty and to encourage integration of computer-based mathematical actions with more traditional methods. Above all, the user must feel "in control" of the software, not controlled by it. The ease with which, for example, the user may return to a previous line and edit the contents rather than retyping demonstrates such a level of control.

The nature of the algebraic learning environment must also bear strongly upon the use of available mathematical tools. The results of this study suggest that, in order to encourage strategic tool use, the environment should be challenging and open-ended. Highly sequenced and predetermined instructional programmes are analogous to teacher-dominated classrooms - they tend to stifle initiative and curiosity, and reward task completion at the expense of enquiry and exploration.

An important result arising from this study concerns the availability of software tools: it appears that such tools can be *too* available under certain circumstances. High incidence of reflexive tool use appeared clearly linked to the *hypertext* design feature in which it was possible to access the graph of any algebraic form encountered in the instructional modules simply by clicking on it. This feature appeared to encourage a superficial viewing of the representation, and, frequently among both students and preservice teachers, an automatic response to moving through the program. In order to encourage more active participation, users should actually enter each algebraic object themselves, and then act upon it in whatever way they choose. As mentioned previously, this **reconstructive** act is likely to force a more analytical viewing of the algebraic object under consideration, and to actively discourage the superficial and passive approach observed commonly in relation to reflexive tool use.

There were other respects, too, in which the design and nature of the computer tools themselves may have contributed in a negative way to student learning and understanding of key mathematical concepts, particularly those associated with the domain and range of functions. The graphing utility developed for the project (based upon a simpler tool created by Dr Khoon Yoong Wong of Murdoch University) was a powerful and versatile package, but was unable to correctly plot discontinuous functions. Thus, single point discontinuities (such as that across the origin in the hyperbola xy = 1) were joined by a line from the bottom of the screen to the top. Graphs of such important functions as $\mathbf{y = log(x)}$ and $\mathbf{y = \sqrt{x}}$, which are undefined for negative values of x, actually plot the value x = 0 along the x-axis in this undefined region. This aspect of the technology was not noticed by the respondents, but may have contributed to subtle misunderstandings. The same misunderstandings may have resulted from the activity in the *Beginning Algebra* module in which the relationships between family members were discussed as an illustration of the function concept. While the relationship “is the wife of” was identified as a function in the mathematical sense, the program failed to draw attention to the important role of domain and range in this context. In particular, it was overlooked that, for the domain in this case (the members of the family), the relationship is undefined for all but one member. Although such an approach to this important mathematical concept is appealing, it is now recognised that it is fraught with dangers and likely to cause confusion and subtle misunderstandings. The tools and the nature of the learning environment itself must be mathematically correct in all respects if they are to be effective in building firm foundations for further study.

Mathematical situations within a technology-rich learning environment should serve to stimulate enquiry and exploration, in addition to discussion and cooperative strategies within social learning contexts.

Van Hiele's third stage of learning, *explicitation*, specifically demands verbalisation as a means towards achieving cognitive progression, in the same way that Vygotsky's theories place social interaction at the heart of effective learning. In this regard, the computer plays a particularly significant role for algebra learning, since it makes explicit both the objects of attention and the processes by which these are acted upon. By making public algebraic thinking and action, mathematical software tools uniquely encourage shared meaning among co-learners, and support insightful evaluation of student thinking and understanding by their teachers.

Finally, a technology-rich algebra learning environment should be characterised by versatile thinking about algebraic ideas using multiple representations, with explicit attention directed towards developing active and meaningful links between these. Such thinking may be encouraged through the thoughtful use of open-ended tasks which appear accessible to the students and yet offer challenges which suggest the use of appropriate tools. For teachers, "learning to ask new questions" remains a critical aspect of "learning to use new tools". The role of the teacher within such an environment must be a flexible one - encouraging individuals to go beyond their present capabilities, and yet not allowing them to become too dependent upon their scaffolding tools. The zone of proximal development remains a concept central to an understanding of such a learning environment, one in which "the only 'good learning' is that which is in advance of development" (Vygotsky, 1978, p. 89) and where it is accepted that "what a child can do with assistance today she will be able to do by herself tomorrow" (Vygotsky, 1978, p. 87). Such a view appears unusual within the context of algebra learning, where traditionally the greatest effort has been placed upon the acquisition of manipulative skills by individuals working alone and unaided. The technology-rich learning environment defined within this study is characterised by two essential features: **challenge** and **support**. The tension between such a view of learning and traditional approaches has led to the identification of a mathematics learning culture which, with aspects of algebraic thinking, may be considered as *intervening* conditions within a theory of mathematical software use.

Intervening Conditions

Both mathematical and pedagogical thinking function as intervening conditions with regard to mathematical software use within an algebra learning situation, potentially impeding or encouraging such use for different individuals.

This study clearly demonstrates that a given algebraic object may be perceived in a variety of ways and associated with a range of action strategies. While simple linear and quadratic equations and graphs signalled predictable responses, simple expressions and tables of values proved more difficult to interpret. Both students and preservice teachers responded to what may be seen as a powerful drive towards closure related to their algebraic actions, a drive frustrated by such a simple expression as **4 - 3x**. When Sfard and Linchevski (1994) asked what individuals see in an expression such as **3(x+5) + 1** (p. 191), they examined the results in terms of Sfard's theory of reification, involving a conceptual move from viewing mathematical concepts as procedures to viewing them as objects, capable of being acted upon in their own right. This study examined simpler objects and found similar perceptions, but focused more closely upon the signal character of such objects and the subsequent repertoire of mathematical actions which they elicited.

Familiarity with graph plotter and table of values served to increase this available repertoire for all participants, offering at least two new strategies for use with what was found to be an impoverished algebraic form. Computer algebra software, however, offered no such addition, merely an alternative approach using traditional methods supported within a computer-based context. The ability of students to integrate the two approaches may be recognised as a determining factor in the use of algebraic manipulation software. This study suggests that such integration may begin early in the formal study of algebra, framed within meaningful context and following upon quite extensive use of the tabular representation. The manipulations of algebra must be grounded in numerical understanding.

The selection and use of available mathematical software tools, then, will be influenced by the algebraic thinking of the user. Algebraic forms which traditionally have signalled a graphical representation (such as the form **y = 2x - 1**) were found in this study to trigger the use of graphing software frequently and spontaneously by all participants. Although the table of values was frequently described as very helpful, it remained a subservient representation to the graph, probably because of difficulties encountered in interpreting tabular information which have been reported elsewhere. Ryan (1993, p. 369-370), citing work by Herscovics (1989) on cognitive obstacles and Yerushalmy (1991) on multiple representational computer software, observes that particular problems arise from the use of multiple representations and the table of values in particular. These include over-reliance upon a single representational form and the assumption that students will naturally and spontaneously “make connections” between different representations. An extensive study by MacGregor and Stacey (1995) involving over 1200 students in two Australian states further supported findings which indicate that tables of values present quite substantial problems of interpretation and analysis. Particular steps need to be taken in order to build effective skills of intrepretation for this representational form (MacGregor and Stacey, 1995, p. 83). At the same time, the table of values was generally considered a valuable aid to understanding within the current study, and perceived by some participants (particularly Stephen and Tony) as a more flexible tool than the graph plotter, capable of acting upon expressions such as **2x - 1** in addition to the more usual form given above.

Stephen's perception of algebraic forms generally appeared to be dominated by an “input/output” or “function machine” metaphor, an active view involving numerical values being changed according to the functional rule. Such an image influenced, not only his interpretation of tables of values (with which this image identifies most readily), but also his thinking about graphs and symbolic forms. The robustness of this active conception may help to explain Stephen’s cross-representational facility and his general success in algebra in comparison with his peer, Ben, who was strongly influenced by a visual graphical metaphor, which did not transfer easily across representations in the same way as did the function machine image. Dependence upon the graphical form alone appeared to disadvantage Ben in his approach to algebraic problems. Although Andrea displayed evidence of both graphical and input/output imagery. her thinking appeared dominated by the symbolic form, especially the equation. It seems likely that her frequent use of the full range of available software tools contributed towards the cross-representational facility she displayed, but her preference for the symbolic form puts her frequent use of computer algebra tools into perspective, in the same way that Ben's strong tendency to visualise helps to explain his use of the graph plotter. Such interaction between thinking and tool use supports the hypothesis of a recursive relationship between the two offered early in this study.

The use of available tools, then, is influenced by the algebraic thinking of the user, and such thinking may come to take the form associated with preferred software tools. Individual perceptions of a given algebraic object and the repertoire of available mathematical actions which it signals may vary greatly, and so determine the nature and direction of tool use. Critical factors appear to be preferred images of algebra, familiarity with the software tools and the degree of integration of traditional algebraic actions with computer-based strategies. Algebraic thinking, however, must be considered in conjunction with pedagogical thinking regarding algebra learning if variations in software use are to be better understood.

In considering the beliefs and perceptions of the participants in this study regarding the nature of algebra, the ways in which it may best be learned and the role of computers in this process, consistent evidence was found to support the notion of a culture of mathematics learning: a shared set of beliefs and experiences which extended across all groups of participants. This culture served largely as an impeding factor for the use of algebra software, characterised as it was by such features as:

- a view of mathematics as “answer-based”, devaluing exploration and open-ended problem solving (those areas in which the software appears most effective);
- a view of algebra as primarily serving a symbolic representation purpose, with little usefulness beyond this role;
- an emphasis upon individual efforts, devaluing both group approaches and the use of external aids (such as computer tools);
- a strong reliance upon individual as opposed to group aids - especially textbooks and hand calculators;
- a dependence upon the teacher as source of knowledge and direction in mathematics learning;
- a limited representational repertoire, dominated by symbolic and graphical forms;
- a lack of reliance upon individual judgement and confidence with regard to their mathematical processes - students appeared quite happy to conclude an answer while expressing little confidence in their result.

Clearly, factors such as these militate against both the *use* and *the perceived need* for open-ended software tools which support and extend mathematical learning. These findings are consistent with the results of other research conducted both in Australia and overseas. Wood and Smith (1993) used a questionnaire adapted from Schoenfeld (1989) to elicit attitudes and beliefs about mathematics from students beginning mathematics and engineering degrees at a New South Wales university. Although such relatively high ability students exhibited positive attitudes and intrinsic motivation with regard to mathematics, threequarters of the sample of seventy four students felt that school mathematics was "mostly facts and procedures that have to be memorised" (p. 594). Alternative approaches to solutions were considered highly desirable (96%) but almost half (44%) felt it was important that mathematics teachers show students "the exact way to answer test questions" (p. 595). Most students felt that a "typical homework problem" should take less than ten minutes (p. 596).

School mathematics in New South Wales is heavily influenced by external examination, and this must exert a powerful effect upon student perceptions and beliefs. A study in progress by Barnes, Clark and Stephens (1995) which compares links between assessment and teaching practices in New South Wales and Victoria has found that teachers in both states value most highly those things associated with high stake assessment. New South Wales teachers (56 teachers in 11 schools) were found to value most strongly "the application of mathematics to real world contexts" and "the use of different mathematical skills in combination" (associated with both School Certificate and Higher School Certificate examinations) and gave low value to "extended and open-ended activities, development of report writing skills, mathematical journals and the provision of substantial written comment on problem solving attempts". The development by students of investigative skills was rated most highly by Victorian teachers and least highly by those in New South Wales.

The powerful influence of mandated assessment upon teaching practice suggested by this study and the previous "ripple effect" study by Clark, Stephens and Wallbridge (1993) suggests a potential role for assessment in encouraging the use of technology in schools. In the short term at least, the use of mathematical software tools with open-ended assessment tasks appears a potentially useful way in which to introduce the use of technology into classroom practice.

At the same time, the mathematics learning culture observed within the current context is clearly not restricted to New South Wales schools. An extensive study by Garet and Mills (1995) involving almost 400 head teachers of mathematics in the United States examined the influence of the National Council of Teachers of Mathematics *Curriculum and Evaluation Standards* upon curriculum content, teaching practices, use of technology and assessment procedures in first-year algebra courses (equivalent to Year 9 in Australia).

> The data indicate that lecture-discussion and in-class problem sets remain the dominant mode of instruction in first-year algebra... the use of calculators has grown dramatically since 1986... The use of computers, although not as extensive as the use of calculators in 1986 and 1991, is expected to increase substantially by 1996... the use of short-answer tests is not declining and remains the dominant form of assessment (p. 382).

Software use was dominated by graph plotters (54% of departments reported using these), drill-and-practice packages (49%) and exploratory packages for algebra and geometry (45%). Only 29% reported use of spreadsheets, and 21% used symbolic manipulation packages. Although cost and hardware factors were considered significant, the strong preference for graph plotting tools and the relatively minor use of computer algebra is supportive of the findings of the current study. Textbooks remain a major influence upon teaching practice.

When considered within the context of earlier studies of exemplary practice in Western Australia and the United States by Tobin and Fraser (1988), the existence of a prevailing culture of mathematics teaching and learning which was so evident within the present study appears irrefutable. The dominant influence of such a culture upon the selection and use of available software tools, then, occupies a central position within the grounded theory of mathematical software use proposed.

Action/Interaction Strategies

The active selection and use of mathematical software tools occupied a central point of focus within this study. Tool-based actions arose in response to the contextual and intervening conditions already considered - the nature and knowledge of available tools, the curricular context and perceptions of the algebra learning environment, and the mathematical thinking elicited by the given situation.

The most frequent mathematical actions for which software tools were used were those associated with graphing (representing), substituting and solving, corresponding to the three main perceptions of the purposes of algebra, as defined by the participants. At a higher level of abstraction, the computer was used most frequently to represent, to verify and, on the part of the tutor, to demonstrate.

Representational actions dominated the computer-based interactions observed within this study. As has already been noted, the graphical form was preferred in most situations, and the students quickly became proficient in the use and interpretation of this software tool. Most effective for this purpose was a “Guess My Rule” game within the IBM-based program *A Graphic Approach to the Calculus*, by David Tall. The computer generated the graph of a selected function type, and students would attempt to identify the particular function. Their guess was graphed, providing immediate feedback. Reluctant at first to make mistakes, the students at all levels quickly overcame this hesitation and learned to use strategic trial-and-error methods to identify different functions. This technique proved so effective that it was incorporated into the *MathPalette* using both graphical and tabular representations. Once again, it encouraged in students familiarity with both representations, and assisted Andrea in particular to become comfortable with the table of values.

Manipulative actions centred upon evaluating substitutions and solving equations. As mentioned previously, the *Theorist* interface encouraged and rewarded both these activities, and a preference was shown by all students for the equation-solving method of moving terms across the equals sign, which was the method supported by this package. While such an approach appears appealing as the preferred method for experienced practitioners, it is also common to find it linked to superficial understanding and rote learning of the solution process, A study by Bell, MacGregor and Stacey (1993) of a group of twenty Year Ten students in a Melbourne school found frequent recourse to what were described as “action memories” in solving simple linear equations. Such memories were tacit, and students were unable to justify their approach. They were also frequently associated with incorrect responses, since the method did not allow for variations in the equation form.

Although the student participants in the present study demonstrated competence in equation solving, both with the computer and without, the method of acting upon both sides of an equation in order to produce a solution would seem pedagogically superior, and computer algebra packages which support this approach preferred. (Note that while *Theorist* does support both methods of equation solving, the manipulation method is so much easier to use that it becomes the preferred option by default.)

As an instructional tool, computer algebra software, then, seems most effective within two situations. The first involves the step-by-step support of extended mathematical processes (such as equation solving in the junior school, “completing the square” in the middle school, and “differentiation by first principles” in the senior years). These processes tend to place high manipulative demands upon students, and so are well-suited to treatment and study using computer algebra tools.

Computer algebra software is also likely to be most effectively used within open-ended mathematical explorations, minimising manipulative barriers and supporting processes of enquiry. As noted within the study, strategic software use occurred most frequently within such a context, in which students were challenged and motivated. Such explorations are not a frequent feature of current mathematics learning situations. However, such use may potentially be encouraged within alternative assessment schemes, in which software tools are made available as a strategic option. The problems gathered and developed for this project are appropriate for use as extended individual and group-based assessment tasks, while actively serving to encourage and demonstrate the use of software tools. As noted in relation to the “ripple effect” studies (Clark, Stephens and Wallbridge, 1993, Barnes, Clark and Stephens, 1995), it is likely to be through such assessment schemes that the use of mathematical software will most readily be integrated into the existing curriculum. Computer tools which make explicit the step-by-step mathematical processes leading towards a solution appear most appropriate for use in open-ended assessment tasks. Word processors have encouraged new approaches to creative writing by making possible a cyclic process involving the refinement of several draft versions. Students using appropriate algebra software may also see their final solution as the end result of an interactive process of refinement, supported by teacher, peers and the software itself.

Consequences

Not all the outcomes which emerged from the use of mathematical software tools in this study were intended. Stephen, for example, believed it necessary for a function to be expressed using a particular format: $f(x) = 2x - 1$ is a function for Stephen, while $2x - 1$ is not. Further, $2x - 1$ cannot be graphed (although it can be represented in tabular form), while $y = 2x - 1$ can. Such misconceptions, while perhaps not serious, arose as a direct consequence of particular features of the software packages which this student had experienced. The use of computer tools with students at all levels, then, must take into account effects such as these. Similarly, Ben's dependence upon the graphical representation must be directly attributed to his encounters with the technology. After becoming comfortable and confident in the use of the graphing tool, Ben became over-dependent upon it.

This project was not intended or designed to establish causality between the use of the software tools and aspects of cognitive functioning by the participants. The experiences of the technology-rich algebra learning environment by both students and preservice teachers were far too limited in relation to traditional mathematical learning contexts to expect clear and attributable changes in skills or knowledge. Even those students who engaged in the research programme for an average of one hour per week over up to two school years spent between four and five times that amount in their usual mathematics classes. It is hardly surprising that the influence of the current mathematics learning culture is so pervasive, and the effects of exposure to an alternative learning environment so few.

Nonetheless, certain consequences of the use of the computer tools could be identified from the data. The evidence of this study indicates strong support for ways in which they may contribute to the learning process, particularly through factors such as:

- increasing confidence in answers (and learning to expect that such higher confidence rates should be the norm rather than the exception);
- increasing the representational repertoire to include tables of values, concrete forms and even animations;
- encouraging exploration and open-ended problem solving by providing tools which facilitate and make possible these approaches.

These three consequences of mathematical software use were widespread among the participants, although varied in degree. Certainly, all student participants indicated improved confidence in their answers as a direct result of the use of computer tools, and all demonstrated some measure of cross-representational facility across symbolic, graphical and tabular forms. All engaged at some stage in strategic use of available tools for the exploration of mathematical ideas. Such use was commonly associated with demonstrations of insight and improved understanding of the mathematical ideas in question.

The responses of all participants reflected positive attitudes towards the use of the computer as an aid to mathematics learning, although most indicated some realistic limitations to their support. Overall, while the graph plotter was enthusiastically accepted as a tool for mathematical learning, the table of values was found to be difficult to interpret at times, and computer algebra software was perceived as being in some ways illegitimate. This is hardly surprising within a culture which convinces students that the “best way to learn algebra” is through repetition, and where the majority of students “prefer to learn the teacher’s method for solving problems”.

A Grounded Theory of Mathematical Software Use

The grounded theory of mathematical software use developed through this study situates such tool use within a context of:

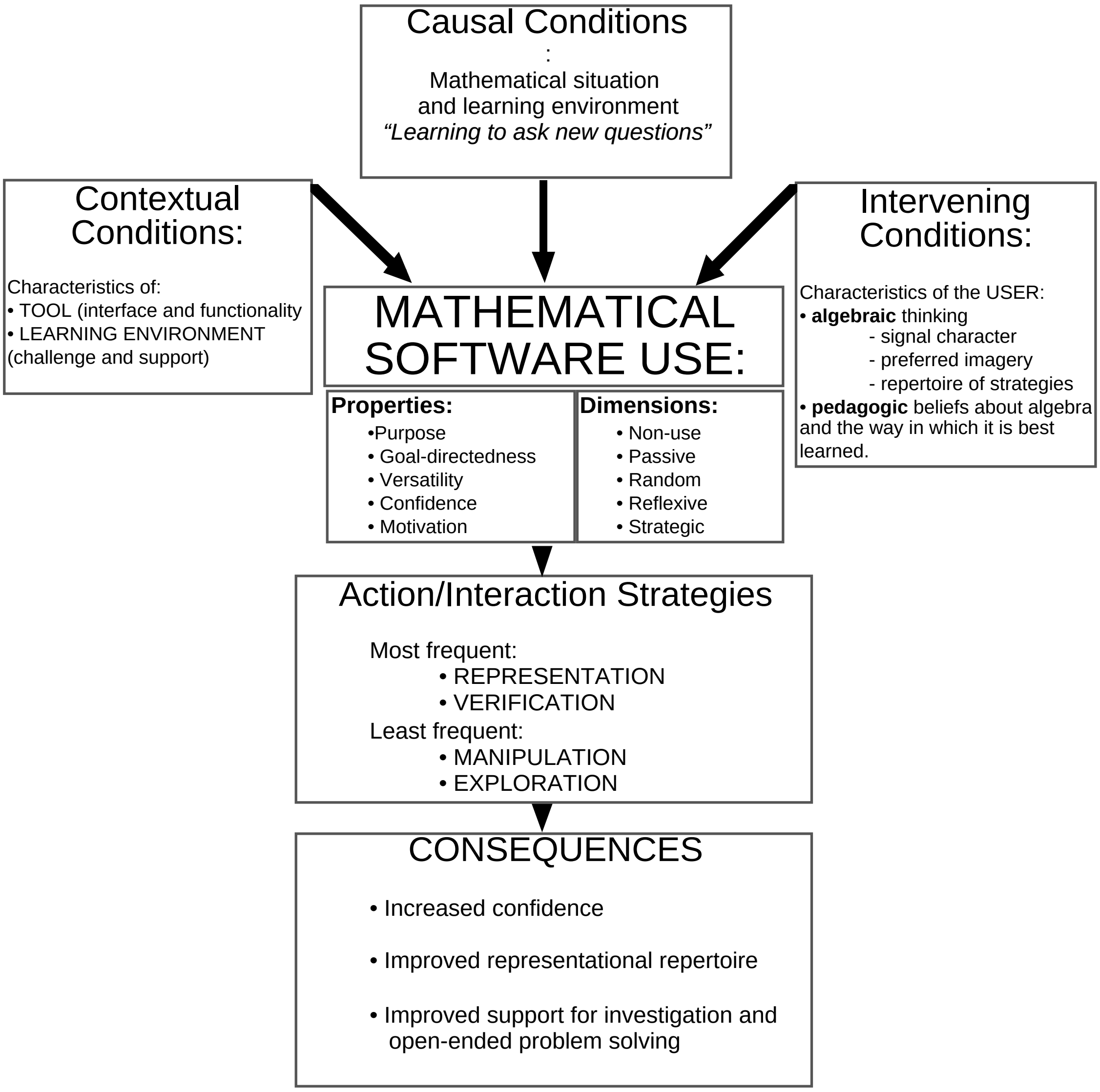
(1) Algebraic and pedagogical thinking by the user,

(2) Familiarity with the software and subsequent access to desired functions, and

(3) A learning environment which is both supportive and challenging, encouraging the use of multiple strategies in order to achieve agreed-upon mathematical goals.

Figure 9.3 provides a schematic outline of the relationship between the various component parts of this theory.

Figure 9.3: A grounded theory of mathematical software use: Schematic outline



The learning environment gives rise to a **mathematical situation**, from which the user must elicit a distinct **algebraic object** (most commonly function, expression, equation, graph or table of values). Subsequent use of mathematical software tools is dependent upon recognition by the student of such an object.

The user seeks to act upon the algebraic object in order to move towards a desired state of closure. The **repertoire of available strategies** is dependent upon both the strength of the signal character of the algebraic object, and upon the extent to which computer-based skills have been integrated with more traditional methods.

The action strategy taken at this stage (which may or may not involve computer use) produces a result which must be evaluated in terms of the extent to which it brings the problem situation closer to closure. While the influence of the prevailing culture of mathematics learning is likely to act against the use of software tools to assist manipulative actions, tool use for purposes of verification of results appears not to inspire the same resistance. Representational tool use, too (especially involving the graphical representation), appears to complement existing practice, while manipulative use directly confronts traditional approaches.

Under conditions of strategic software use, the evaluative act is likely to present to the user a new mathematical situation, requiring further interpretation, action and reflection. Strategic use is characterised by persistence, curiosity and the use of multiple strategies for both exploration and evaluation of results. It frequently accesses multiple representations and uses a range of available software tools. Such use is goal-directed, flexible and frequently insightful.

Under conditions of high availability of tools, high demand for task closure and extrinsic motivation, the use of available tools is likely to be **reflexive**, as the user selects quickly and makes superficial use of desired functions. Such use is commonly associated with use of the graph plotter as the preferred tool of choice and an environment which rewards results rather than process. Reflexive tool use may be discouraged by requiring active involvement on the part of the student, particularly in the reconstruction and entry of algebraic forms. Students must be participants, not observers, in the mathematical process.

Under conditions of free orientation, **random** tool use may occur, as the user experiments with available tools. Such use may be versatile, but it is low in goal-directedness. **Passive** tool use occurs under conditions of an imbalance of power between multiple users. While such use commonly involves teacher (or tutor) and students, it may also be observed between peers working together. The user in such a context hands over the responsibility for learning to the active participant, and is likely to gain much less from the experience than would an active participant. This role is associated with the use of the computer as a tool for demonstration (Ganguli, 1992).

Finally, there are conditions under which tool use is appropriate, and yet no such action is taken. It was common for participants to express lack of confidence in their answers, and yet to take no action to validate or disprove their result. Students appeared to feel no personal commitment regarding their involvement in algebra learning: motivation is extrinsic and the demand for closure apparently far exceeds interest or curiosity regarding the mathematical situation.

Within the constraints of the research design, the grounded theory proposed offers the possibility of prediction regarding the likely use of mathematical software tools and the encouragement of strategic use within algebra learning situations.

The principal uses of computer tools for mathematical purposes within this study were found to be for **representation** (using graph plotter and, less often, table of values) and **verification** of results. Although well-suited to support open-ended investigation, such use is likely to remain rare under the influence of a culture of learning which rewards closure and identifies algebra with “finding an answer” using automated and predetermined action sequences. Consequently, computer algebra tools may best be introduced into the current mathematics curriculum in two ways:

- As means of supporting students in the learning of sequential mathematical procedures (such as equation solving in the early years). Computer tools which both support and make explicit the process provide a useful aid in such areas.
- As tools for supporting open-ended assessment tasks, and so encouraging and motivating mathematical enquiry.

The evidence of this study suggests that teachers may encourage strategic software use through the creation of a learning environment within which:

- *students are comfortable with the available software tools*. The interface should support ease of entry of mathematical forms and make the range of mathematical functions clearly available.
- *mathematical tasks lie within the zone of proximal development of the students*. Students must perceive the task as potentially

achievable, although beyond their present capabilities unaided.

- *students must be able to elicit from the task a mathematical object* which is capable of signalling appropriate action strategies involving the integration of mathematical and computer-based actions.
- *open-ended investigation is perceived as a valid means of achieving a solution*, which may be only one of several appropriate responses to the task.
- *The use of multiple strategies for verification must be perceived as a necessary component of mathematical enquiry.*
- *students must be motivated*: persistence and some measure of personal commitment to the solution process must be evident.

The strategic use of mathematical software tools is indicative, not only of a high level of computer-based competence, but of insightful and strongly connected mathematical thinking. Conditions under which such use may be encouraged should be a feature common to all mathematics learning situations.

Conclusion: Impediments, Imperatives and Implications

The grounded theory proposed appears both dense and integrated. As a teacher learning to use new tools, I feel confident that the initial demands of my action research enquiry have been satisfied. The phenomenon has been examined in great detail, and situated within a broader context: one involving images and definitions of algebra, perceptions and beliefs about learning, recognition of the characteristics of “good” algebra software and some appreciation of what a “technology-rich algebra learning environment" may look like. As the researcher and prime motivator for this study, I feel confident in my new knowledge and skills regarding "teaching with these new tools". Always there is more to know, but I recognise that at least now this teacher knows enough to "get started".

One of the most informative features of the study involved recognising the formidable array of impediments to the use of mathematical software tools for algebra learning. While readily recognising what might be termed "institutional" constraints (particularly lack of access to appropriate hardware and software) this study made it obvious that the real impediments were buried deeper, within the psyche of mathematics teaching as it has been practised in our society for one hundred and fifty years. This impediment will be difficult to overcome, since it arises from perceptions of the very nature of algebra, as it is found in schools.

For every impediment associated with the use of computer technology in schools, there are a growing number of imperatives. From the demands of society for a technologically-literate and mathematically competent work force to the surprising wonders of chaos theory, the symbiotic disciplines of mathematics and computing will continue to cross paths again and again. As computer technology becomes ever more accessible, appropriate and powerful as an environment for learning, the possibilities it offers for improved understanding of mathematical ideas and support for mathematical skill development become impossible to ignore.

This study may be seen to have implications for a variety of audiences with an interest in the role of technology in mathematics learning. For those interested in examining further the issues related to this critical field of enquiry, the theory proposed here suggests many new questions. At an individual level these include the effects of different computer algebra tools upon the development of manipulative skills and mathematical understanding, a more detailed examination of the role of preferred imagery in algebra learning and its relationship with action strategies, the transfer of meaning across representations and the derivation of meaning from new representations made possible by computer technology.

With the increasing power and availability of hand-held computers and graphic calculators, issues of personal access to technology must also be addressed. Students in the current study used their own calculators effectively and often. As Smith (1992) proposed, the distinction between a *Social Constructivism model* of tool use (in which the tool and the user act jointly upon the mathematics “out there”) as opposed to an *Individual Constructivism model* (in which both mathematics and tool are “out there”) is relevant here. The calculator fitted comfortably within the “personal space” of the user; the computer did not. In this study, it remained “out there” and students failed to achieve the comfortable and spontaneous familiarity with the computer tools which distinguished their calculator use, even after protracted experience. It remains to be seen whether hand-held computer tools capable of supporting algebraic manipulation and multiple representations may be more readily accepted and utilised.

At a classroom level, factors influencing the classroom use of computer tools for mathematics learning must be considered, especially the role of the computer in assisting group and cooperative approaches by making public both algebraic objects and processes. The physical impediments associated with access to the technology for large groups and the changing role of the teacher within a technologically-rich learning environment are increasingly important considerations. Asp, Dowsey and Stacey noted that teachers moved from an instructional to a management role and tended to miss capitalising upon learning opportunities as a result (Asp, Dowsey and Stacey, 1993, p. 53). At both classroom and individual levels, the responses of teachers to technology remain of critical importance. The decision to restrict the present study to those engaged in algebra learning was a deliberate one, allowing a necessary restriction of focus which would not have been possible otherwise. Nonetheless, the next logical step from this study must be an examination of the individual interactions of teachers with the technology, and a close examination of the nature and influence of their algebraic and pedagogical thinking upon tool use. Such a study offers much in deepening and potentially verifying the present grounded theory of mathematical software use.

Broader issues still relate to the influence of technology upon the nature of learning and instruction. Trying to adapt the use of the technology to fit the existing mathematics classroom may well be a retrograde step: as demonstrated forcibly in this study, there are fundamental incompatibilities between the effective use of technological tools and the prevailing culture of mathematics learning and instruction, at least as evident within this sample. If such tools are to be used to their full advantage, then critical beliefs and assumptions about the nature of algebra and the ways in which it is best learned, and even of what constitutes effective learning and successful teaching must be revised and, perhaps, sacrificed.

For mathematical software developers, this study provides detailed information by which software tools for mathematics learning may be evaluated. This study provided an opportunity to gather, use and evaluate a large collection of mathematical software tools. Very few of those currently available satisfy the criteria developed through interaction with the participants. There remain opportunities for the creation of appropriate mathematical tools which support and encourage mathematical enquiry, and potentially offer access to much of mathematics which is interesting, relevant and important, but is currently denied to the majority of students who "do not possess adequate algebra skills".

For teachers of mathematics, this study provides an opportunity to share in the learning experience of a colleague. Before it is possible teach effectively, it is necessary to have some understanding of the ways in which individuals learn, and this study specifically provides such information with regard, not only to the use of computer software tools, but also in relation to the ways in which individuals perceive and act upon algebraic forms, the influence of preferred algebraic imagery, and beliefs and perceptions regarding learning. Further, the computer-based instructional modules developed, trialed and evaluated within this project provide practical ways in which teachers and students may explore the use of computer tools for the learning of algebra. For teacher educators, too, the modules provide a simulated algebra learning environment in which preservice teachers may examine alternative approaches to algebra teaching and learning made possible through the use of software tools.

This study offers powerful arguments for the use of appropriate computer technology in the creation of an algebra learning environment which emphasises meaningful contexts, the development of versatile thinking about algebra through multiple representations, and a balance between challenge and support. Mathematical software tools offer unique opportunities for the development of algebra skills within a context of improved understanding and active involvement by students in their own learning. By making explicit both algebraic thinking and processes, appropriate software encourages feedback, verbalisation and cooperative approaches on the part of the learners, and supports informed evaluation by the teacher.

Most important, however, is the possibility for exploration of mathematical ideas supported and made possible by such tools. The view of mathematics which follows from such an approach is one which is vibrant and exciting. Not only are learners placed in positions of responsibility and control regarding their own learning, and the likelihood of learning with true understanding substantially heightened, but, for perhaps the first time, teachers and students potentially become co-learners in stretching the boundaries of their discipline.

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