The Beach Race

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Grade level: secondary (Years 9-12)
Subject: mathematics
Time required: 90 minutes

Activity overview

This beach race begins from a point 4 km out to sea from one end of a 6 km beach, and finishes at the opposite end. Contestants must swim to a point along the beach, and then run to reach the finish line first. I can swim at 4 km/h and run at 10 km/h – where should I aim to land on the beach so as to minimize my total time for the race?

Background

This problem has a distinctly Australian flavour, and has long been popular as an optimisation problem involving differential calculus in senior years. Using appropriate technological tools, however, it is also accessible as a valuable algebraic modeling experience for a much wider range of students.

Concepts

Pythagoras' Theorem, algebraic modeling, maxima and minima using calculus

Teacher preparation

Some discussion of the nature of mathematical modeling and the key concepts of variable and domain would be valuable in introducing this activity. Students should expect to move from a geometric model to a numerical one; from numbers to graph, and finally derive an algebraic expression, which may be verified using the graph previously derived. The purpose of this algebraic model should be clear to students – it allows much closer investigation of the situation, and should support meaningful predictions on the part of the students.

For this task, some specific discussion of the relationship between distance, speed and time will be necessary.

Classroom management tips

Some introductory teacher-led discussion, making explicit early interpretations of the task, is very important. Students should always be encouraged to articulate what they expect to happen before they begin using technological tools. Pen and paper sketches of the problem should lead quickly to interactive geometric models capable of generating data points.

It is always valuable to begin a challenging activity with students working for at least a few minutes individually (in order to engender some attempt and some measure of commitment to the task) and then encouraging students to work in pairs. Later, pairs could join into groups of four in order to provide further stimulus for discussion, and further support for those students who might be struggling.

Technical prerequisites

Students should know how to:

- *Construct simple geometric models using segments and variable points, and measure the lengths and distances relevant to the task (**Graphs & Geometry** application).*
- *Use the **Calculator** application to substitute measured values into an appropriate formula (entered using the Text tool) and use the **Lists & Spreadsheet** application to store these values for further study (**Graphs & Geometry** application, **Calculator** application and **Lists & Spreadsheet** application).*
- *Store numerical values as variables in order to build an algebraic model which forms the basis of prediction (**Graphs & Geometry** application, **Calculator** application and **Lists & Spreadsheet** application).*

Step-by-step directions

1. Introductory discussion could centre on the relationship between distance, speed and time, with students providing examples from their own experience (for example, traveling 20 km to school should take 20 minutes at an average speed of 60 km/h).

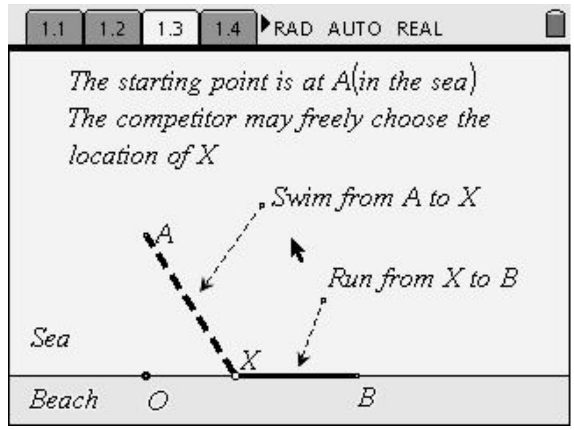
   Personal attributes should also be featured in this discussion – how fast do you walk? How fast do you think you can swim? If time allows, use of a CBR2 or other suitable motion detector would support and motivate students in this, as well as building strong understanding of the factors involved in this task.

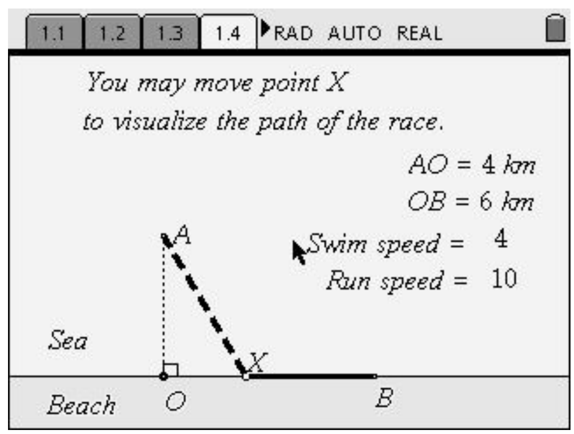
   The context of the Beach Race may then be introduced, and some time for student discussion allowed. Either individually or in pairs, students should be encouraged to sketch a pen-and-paper diagram of the problem situation as they understand it.

STAGE 1: INTRODUCTORY DISCUSSION and sharing of ideas.
20 minutes

STAGE 2: Graphs & Geometry application modelling (20 minutes)

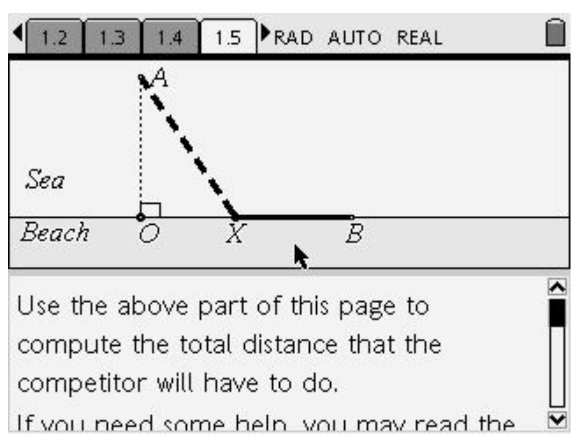
2. The introduction of the Graphs & Geometry application should be seen as a natural extension of students' previous pen-and-paper modeling. Individually or in pairs, students should construct their own models of the beach race, and begin to take relevant measurements. Some class discussion may be needed at regular intervals, allowing students to share understandings and techniques with others who may need additional support.





STAGE 3: Building a TOTAL TIME formula [10 minutes]

3. The key concept of TOTAL TIME for the race needs to be clear to all students – this is the sum of the time for the swim and the time for the run.



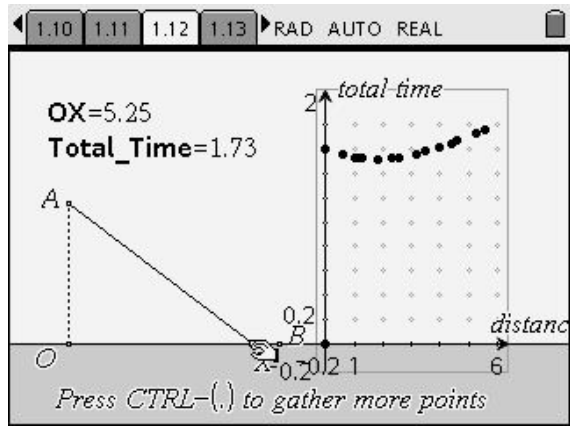
4. It would be helpful to begin with an exploration of TOTAL DISTANCE using the model provided, and lead from this to the idea of TOTAL TIME.

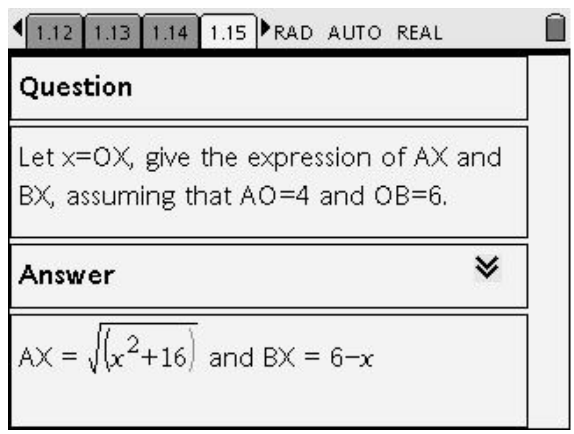
5. A need for greater precision should arise naturally from class discussion, and give rise to the need for an algebraic model, supporting prediction and more involved calculation (including finding minimum values through use of calculus).

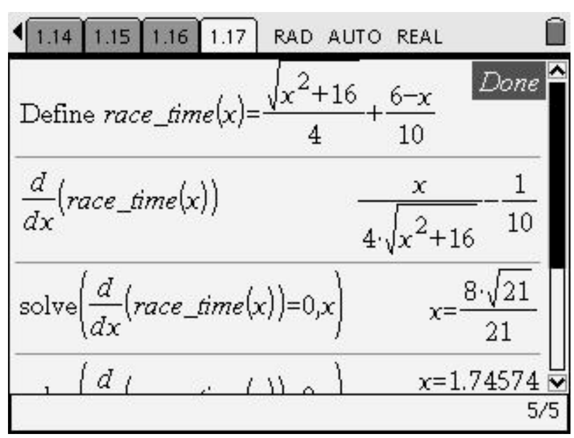
   The graph of the algebraic model, of course, is a powerful tool for verification when compared with that of the geometric model. Opportunities clearly exist for a CAS extension to this problem, using the symbolic tools of calculus offered by this platform. However, the problem may also be attempted using numeric tools (numerical solve, numerical derivative or numerical function minimum).

STAGE 4: Interpreting results and drawing conclusions [10 minutes]

STAGE 5: Building and Verifying an Algebraic model [30 minutes]







Assessment and evaluation

The start of the beach race is a buoy, located 4 km from the end of the beach. The finish lies at the opposite end of the 6 km beach.

1. Find the total length of the race if I swim the shortest possible distance?

   4 km (from start to beach) + 6 km (length of the beach) = 10 km.

2. Find the total length of the race if I swim the entire distance to the finishing line?

   $d = \sqrt{4^2 + 6^2} = \sqrt{52} \approx 7.2km$

3. Which method would take the longer time, and how much longer?

   Option 1 would take approximately 1.6 hours, the second 1.8 hours, and so would be about 20 minutes longer.

4. What landing point would lead to the longest time for the race?

   The last answer would be the maximum time for the race (1.8 hours)

5. Find, as accurately as possible, the x-value that would lead to the minimum time?

   X=1.75 using the graph, 1.74574312189 using algebra and calculus!

6. What are the main limitations of this model? What assumptions have you made in creating this model?

   Some of the many limitations include the accuracy of the construction, as well as neglecting “reality factors” – maintaining the same speed under all conditions, swimming in perfectly straight lines, and assuming the beach to be perfectly flat.

Students should be expected to document their solution carefully, detailing in their own words the meaning of each result, and the reasons for their choice of approach. In particular, they should be able to clearly describe both the strengths and weaknesses of their models.

Teachers should ensure that they recognize and reward elements of persistence, care in construction and measurement, and ability to justify reasoning in responses submitted for this task.

Activity extensions

- Students should consider other approaches which may be possible for this problem – in particular, what might be reasonable values for the transition between surf and sand: for what length of time would racers be somewhere between their swim speed and their run speed as they reach shore? How might this be built into a model?

- Is it possible to generalize the solution to this problem? Is there an optimal relationship between swim speed and run speed that might influence the result?

- Other types of races could be examined in this way – for example, triathlons are popular events involving swimming, running and cycling. A suitable extension assignment could involve students in modeling a triathlon event, using their own measurement and other sources (such as the Internet) to derive reasonable estimates of speeds for each leg.

