Cobweb Graphs: Investigating Chaos

ID: XXXX

Time required
45 minutes

Activity Overview

In this activity, students investigate population growth using a variety of representations, leading to some observations about stable and unstable populations as an introduction to chaos theory.

Concepts

- *Iterative processes, sequences, graphical representations of discrete data.*

Teacher Preparation

This investigation offers opportunities for students to explore new branches of mathematics using multiple representations and a variety of mathematical tools. As such, care should be taken to provide ample time for ALL students to engage actively with the requirements of the task.

- *This activity can serve to introduce work on sequences and iteration.*
- *Begin by discussing how mathematics is a growing branch of human knowledge and that much remains yet to be explored.*
- *The screenshots on pages X–X (top) demonstrate expected student results. Refer to the screenshots on page X (bottom) for a preview of the student .tns file.*
- ***To download the .tns file, go to http://education.ti.com/exchange and enter "XXXX" in the search box.***

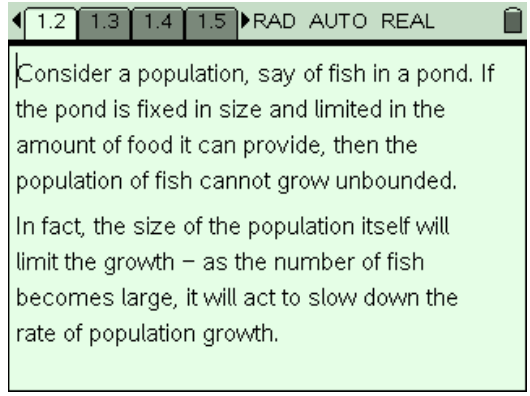
Classroom Management

- *This activity is intended to be **student led.** You should seat your students in pairs so they can work cooperatively on their handhelds. Use the following pages to present the material to the class and encourage discussion. Students will follow along using their handhelds, although the majority of the ideas and concepts may be fully presented in **this** document; be sure to cover all the material necessary for students' total comprehension.*
- *Students can either record their answers on the handheld or you may wish to have the class record their answers on a separate sheet of paper.*

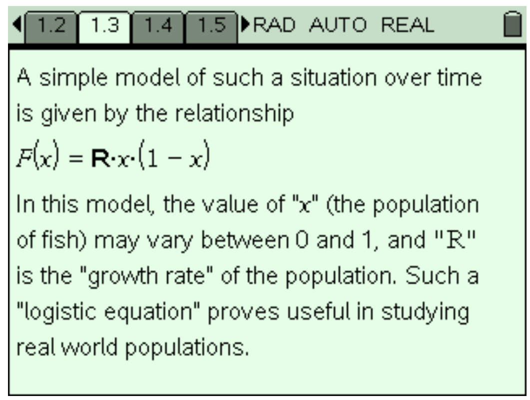
TI-Nspire™ Applications

Calculator, Graphs & Geometry, Lists & Spreadsheet, Data & Statistics, Notes.

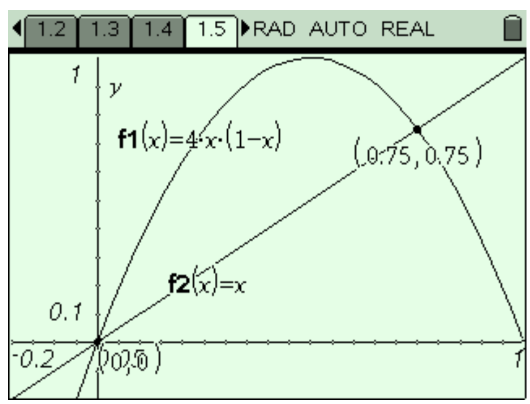
Step 1: Begin with discussion concerning applications of mathematics to such important areas as ecology and population growth, with mention of the new field of chaos theory as helping us to understand stable and unstable growth in complex systems.



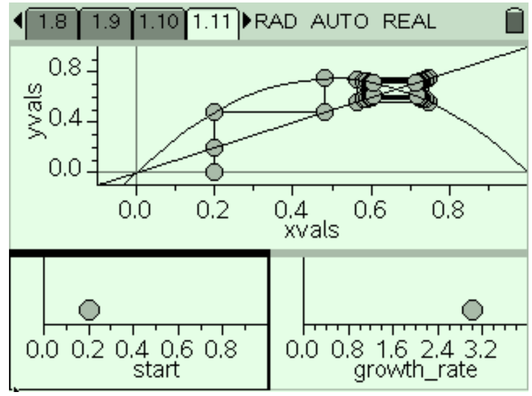
Step 2: Students should understand the way in which the simple quadratic relationship shown describes a model of growth for many naturally occurring populations.



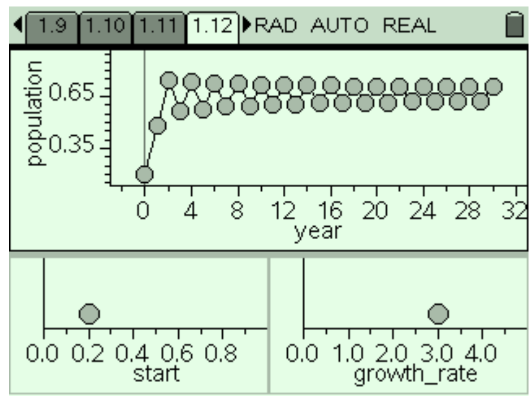
Step 3: In particular, students should be led to an appreciation of the role of equilibrium points as describing population values which may be expected to lead to stability: if x is the initial population, and the function F(x) describes the growth pattern, then the population after one year will be F(x), and after two years, will be F(F(x)), and so on. Now if F(x) = x, then, F(F(x)) = x, and so on for subsequent years.



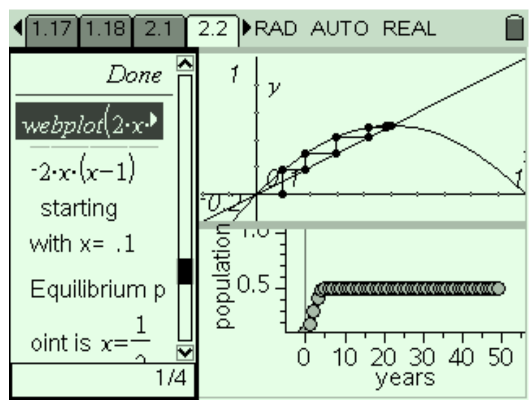
Step 4: More interesting still is to consider the behavior (or "orbit") of a point which is NOT an equlibrium point, and how this changes over several years. A CobWeb Graph or webplot as shown offers a useful representation by which students may explore the impact of changes to both the starting value and the growth rate upon the long-term population growth.



Step 5: Students will quickly observe that the actual starting value has far less impact upon population than the variable, R, which describes the population growth. Some values of R lead to regular, predictable cycles of growth, others to chaotic unstable behavior. Students should use the tools available to explore and attempt to locate the critical values of R for such occurrences. Introduction to the ideas of period doubling is important here.



Step 6 (CAS EXTENSION): The availability of a CAS-based program for quickly and easily varying conditions and even the functions involved opens the doors for further exploration: what sort of functions lead to predictable growth patterns, and what sort lead to chaos?



Students should begin to appreciate that this part of mathematics is new and growing – unlike so much of the mathematics studied in schools. There are opportunities for discoveries and surprising new knowledge to be gained by those willing to be both patient and systematic in their explorations!

CobWeb Graphs: Investigating Chaos – ID: XXXX

(Student)TI-Nspire File: *ActXX_Cobweb_Plots_EN.tns*

Cobweb Plots:
Investigating Chaos

Consider a population, say of fish in a pond. If the pond is fixed in size and limited in the amount of food it can provide, then the population of fish cannot grow unbounded.

In fact, the size of the population itself will limit the growth – as the number of fish becomes large, it will act to slow down the rate of population growth.

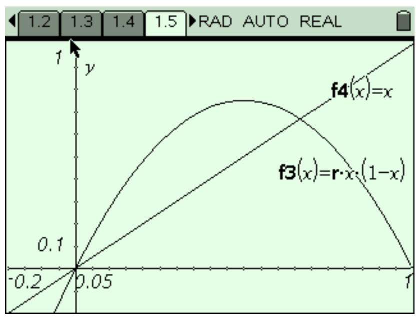
A simple model of such a situation over time is given by the relationship

$F(x) = \mathbf{R} \cdot x \cdot (1 - x)$

In this model, the value of "x" (the population of fish) may vary between 0 and 1, and "R" is the "growth rate" of the population. Such a "logistic equation" proves useful in studying real world populations.

Note that as x approaches its upper value of 1, this will cause the population to decrease. The points where the graph of this function meets the identity function $(I(x) = x)$ are called **equilibrium points** and indicate zero population growth in a year.

Graph $y = 4 \cdot x \cdot (1-x)$ and $y = x$ and find their intersection points.



In the example shown, we see that there are two equilibrium points, at $x = 0$ and $x = 0.75$.

These are the solutions to the equation $4x \cdot (1 - x) = x$ (where R = 4).

Question

1. Can you explain why such equilibrium points indicate zero population growth?

Answer

More interesting is to consider the behaviour (or **orbit**) of a point which is NOT an equilibrium point.

For varying values of R, the behaviour of such orbits is quite different.

Use the spreadsheet and graphs that follow to explore the effects of varying the initial population, (**start**) and the growth rate (**R**).

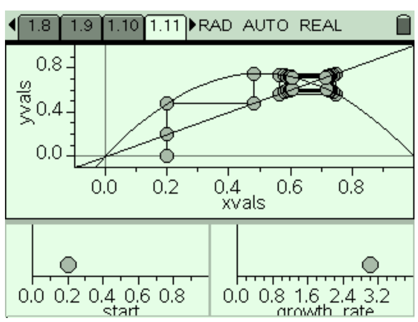
	A year	B popul...	C start	D growth...
◆	=seq(x,x	=seqn(d1*u		
1	0	.2	.2	3
2	1	.48		
3	2	.7488		
4	3	.564296		
5	4	.737598		

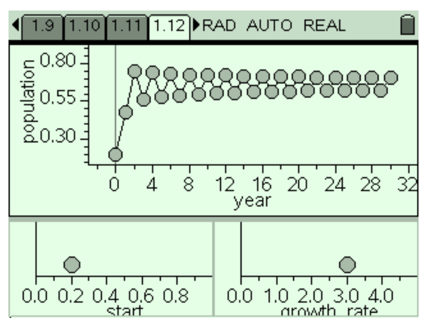
A1 =0

On the next page, you will see an example of a web plot or cobweb diagram, showing the orbit of the start point over many generations.

Another way to view this is to actually observe the population itself changing over time (the following page).

For each, vary the start and growth rate and observe the effects.





1.11 1.12 1.13 1.14 RAD AUTO REAL

Question

2. Carefully describe the effects upon the population growth of changes to the **start value** .

Answer

1.11 1.12 1.13 1.14 RAD AUTO REAL

Question

3. Carefully describe the effect of changing values of the **growth rate** upon the population growth.

Answer

1.12 1.13 1.14 1.15 RAD AUTO REAL

One of the characteristics of chaotic behaviour in complex systems appears to be a feature known as period doubling.

When R = 3, you should observe that the population oscillates between two stable values – this is an example of a period 2 cycle.

1.13 1.14 1.15 1.16 RAD AUTO REAL

By the time R = 3.5, however, there is clearly a change – in fact, we have moved to a period 4 cycle (hence the term "period doubling"!)

1.14 1.15 1.16 1.17 RAD AUTO REAL

Question

4. Try to find the values of R at which these period changes occur, from 2 to 4, from 4 to 8...

Answer

1.15 1.16 1.17 1.18 RAD AUTO REAL

By around 3.83... all cycles of the form 2^n appear to have been exhausted, and 3–cycles take over.

These occur even more rapidly and by the time R = 4 the system has reached a chaotic state.

1.16 1.17 1.18 2.1 RAD AUTO REAL

CAS Extension

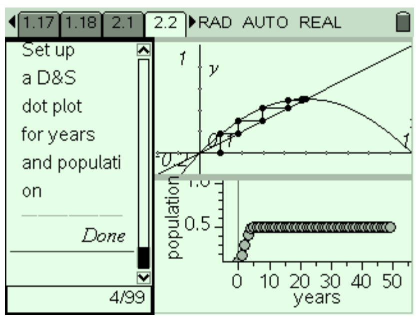
If you have access to a computer algebra system, you may explore these elements of chaos further.

Use the program

webplot(function, start, iterations)

to study other functions in this way.

e.g. try $2x$ and even $\sin(\pi \cdot x)$.



1.18 2.1 2.2 2.3 RAD AUTO REAL

	A xval	B yval	C years	D popula...
◆				
1	.1	0	0	.1
2	.1	.18	1	.18
3	.18	.18	2	.2952
4	.18	.2952	3	.416114
5	.2952	.2952	4	.485926

A1 .1

2.1 2.2 2.3 2.4 RAD AUTO REAL

Question

5. Can you find other functions that lead to equilibrium or to chaos?

Answer

2.2 2.3 2.4 2.5 RAD AUTO REAL

These are the elements of chaos, a field at the cutting edge of mathematics and human knowledge. There is much yet to be learned about its properties and applications, and yet it is accessible through such simple models as those studied here. Be patient and systematic and you may discover something new and unexpected!

