

Introducing the Differential Calculus From First Principles

Time required
45 minutes

ID: XXXX

Activity Overview

In this activity, students begin with the concept of gradient between two points, which lie on the graph of a function. They learn that for many functions the gradient (or rate of change) actually changes constantly and we can calculate this rate at every point of a continuous function. Students begin with a graphical treatment and move on to an algebraic approach.

Concepts

- Gradient at a point as the limiting value of a segment as the segment length approaches zero.
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Teacher Preparation

This investigation offers opportunities for review and consolidation of key concepts related to the gradient and functions, and introduces essential understanding of limit as a value approaches zero. As such, care should be taken to provide ample time for ALL students to engage actively with the requirements of the task, allowing some who may have missed aspects of earlier work the opportunity to build new and deeper understanding.

- This activity can serve to consolidate earlier work on gradient. It offers a suitable introduction to limit and instantaneous rates of change.
- Begin by discussing how most changeable quantities vary constantly making an average treatment quite limited. Introduce the idea of instantaneous rates of change.
- The screenshots on pages X–X (top) demonstrate expected student results. Refer to the screenshots on page X (bottom) for a preview of the student .tns file.
- To download the .tns file, go to <http://education.ti.com/exchange> and enter "XXXX" in the search box.

Classroom Management

- This activity is intended to be **teacher led**. You should seat your students in pairs so they can work cooperatively on their handhelds. Use the following pages to present the material to the class and encourage discussion. Students will follow along using their handhelds, although the majority of the ideas and concepts are only presented in **this** document; be sure to cover all the material necessary for students' total comprehension.
- Students can either record their answers on the handheld or you may wish to have the class record their answers on a separate sheet of paper.

TI-Nspire™ Applications

Graphs & Geometry, Notes, Lists & Spreadsheet.

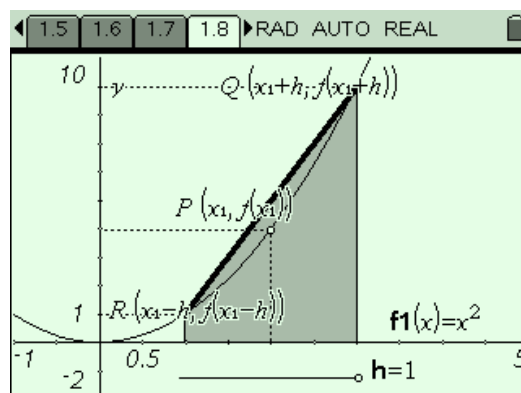
Step 1: Begin with discussion concerning common experiences of average rate of change compared with instantaneous rate of change, such as the time taken for a car journey. If necessary, review the ideas and skills associated with finding gradient between two points.

1.1 1.2 1.3 1.4 ▶RAD AUTO REAL

Begin with a function $f_1(x)$ describing my journey and a point $P(x_1, f_1(x_1))$ which lies on the graph of the function, $y = f_1(x)$.

Now picture another point, Q , which also lies on the graph of $y = f_1(x)$, close to P . In fact, suppose the distance from P to Q along the x -axis is h units, where h is a small amount.

Step 2: Carefully move students from the general gradient form to the gradient between the two points $Q(x_1+h, f(x_1+h))$ and $R(x_1-h, f(x_1-h))h$ should be carefully considered.



Step 3: It may be helpful for students to return to the graph on page 1.8 and view the graph of the gradient function (already defined as $f_2(x)$) – show the graph entry line if hidden (CTRL-G) and press ENTER after $f_2(x)$. Observe the changes in the graph of the gradient function as h changes.

1.10 1.11 1.12 1.13 ▶RAD AUTO REAL

The differential calculus defines the gradient function in terms of a limiting value as the value of h approaches zero. i.e.

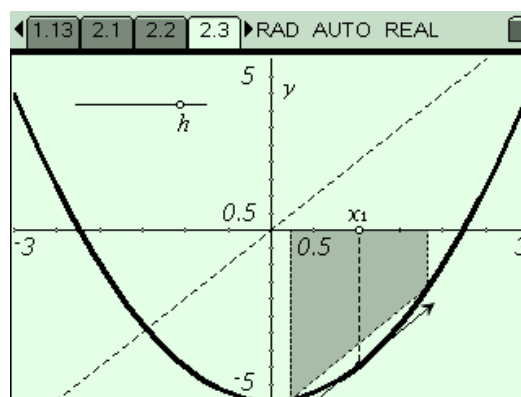
$$grad_{fn}(x) = \lim_{h \rightarrow 0} \left(\frac{f(x+h) - f(x-h)}{(x+h) - (x-h)} \right)$$

$$grad_{fn}(x) = \lim_{h \rightarrow 0} \left(\frac{f(x+h) - f(x-h)}{2h} \right)$$

Step 4: (CAS Extensions) The dynamic algebra facility available within **Lists & Spreadsheet** application is well-suited to this type of algebraic process. Students enter a function of their choice and follow the step-by-step process leading to the gradient function evaluated in cell B5. They may use this template to set out their own solutions for other functions, at first using this tool as a scaffold, and then doing their own work and checking their solutions against the spreadsheet.

	A	B	C
1	y =	$x^2 - 5$	
2	P(x, y)	x	$x^2 - 5$
3	Q(x, y)	x+h	$x^2 + 2hx + h^2 - 5$
4	R(x, y)	x-h	$x^2 - 2hx + h^2 - 5$
5	GradQR	2*x	
A1 "y ="			

Step 5: Once again, the graphical representation should be studied as a support for students to begin to generalize what they have learned.



Step 6 (EXTENSION): Further scaffolding is provided in the form of a program, **FirstPrinciples(function)**, which carefully lays out the solutions sequence for any function entered. Students may use this to verify their own solutions.

Finally, a challenge may be set: are there functions for which such methods do NOT work? While students may not be able to find actual functions which are not differentiable, they should come to appreciate that there exist many functions which are not differentiable for all points in their domain.

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firstprinciples(x·ln(x))
f(x) = x·ln(x)
Begin with a point on the curve:
P(x1, f(x1)) = {x1, x1·ln(x1)}
Now consider other points close to P
Q(x1+h, f(x1+h)) = {h+x1, (h+x1)·ln(h+x1)}
R(x1-h, f(x1-h)) =
    
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Introducing the Differential Calculus from First Principles – ID: XXXX

(Student)TI-Nspire File: *CalcActXX_First_Principles_EN.tns*

1.1 1.2 1.3 1.4 ▶RAD AUTO REAL

Introducing the Differential Calculus from First Principles

CALCULUS

1.1 1.2 1.3 1.4 ▶RAD AUTO REAL

My trip to school each day takes me 20 minutes to travel 15 miles. We would say that my average speed for this trip is 45 miles per hour.

In reality, however, I speed up and slow down continuously over that time. The differential calculus allows us to describe instantaneous rates of change rather than just averages!

1.1 1.2 1.3 1.4 ▶RAD AUTO REAL

In this activity, we begin with the concept of gradient between two points, which lie on the graph of a function. We learn that for many functions (like my trip to school) the gradient (or rate of change) actually changes constantly and we can calculate this rate at every point of my journey, not just at start and finish.

1.1 1.2 1.3 1.4 ▶RAD AUTO REAL

Begin with a function $f_1(x)$ describing my journey and a point $P(x_1, f_1(x_1))$ which lies on the graph of the function, $y = f_1(x)$.

Now picture another point, Q, which also lies on the graph of $y = f_1(x)$, close to P. In fact, suppose the distance from P to Q along the x-axis is h units, where h is a small amount.

1.3 1.4 1.5 1.6 ▶RAD AUTO REAL

Question

1. Give the coordinates of the point Q in terms of x_1 and the function $f_1(x)$.

Answer

1.3 1.4 1.5 1.6 ▶RAD AUTO REAL

Question

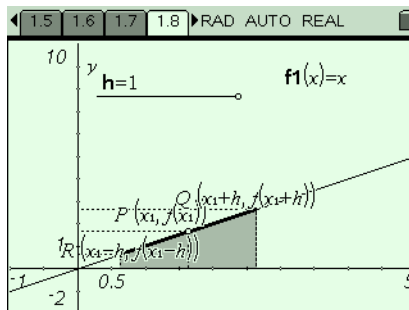
2. Another point R lies the same distance on the other side of P. Give the coordinates of the point R in terms of x_1 and the function $f_1(x)$.

Answer

1.4 1.5 1.6 1.7 ▶RAD AUTO REAL

We are interested in the slope or gradient of the segment QR for different functions, for different points along these functions, and for different values of h .

On the next page study the slope of QR, first for $f_1(x) = x$ and then for $f_1(x) = x^2$. Drag the point P and then change the size of h .



1.6 1.7 1.8 1.9 ▶RAD AUTO REAL

Question

3. What do you observe about the slope of QR for linear and non-linear functions?

Answer

1.7 1.8 1.9 1.10 ▶RAD AUTO REAL

Question

4. How would you calculate the gradient of the segment QR for $Q(x_2, y_2)$ and $R(x_3, y_3)$?

Answer

1.8 1.9 1.10 1.11 ▶RAD AUTO REAL

Question

5. Now express this gradient formula using the relationships $x_2 = x_1 + h$, $y_2 = f(x_1 + h)$, $x_3 = x_1 - h$, and $y_3 = f(x_1 - h)$.

Answer

1.9 1.10 1.11 1.12 ▶RAD AUTO REAL

It appears that for non-linear functions, the gradient of the segment PQ changes for different values of x_1 and for different values of h .

Return to page 1.8 and graph the gradient function you have just calculated. Carefully observe the effect of changes in the value of h upon this graph.

1.10 1.11 1.12 1.13 ▸RAD AUTO REAL

The differential calculus defines the gradient function in terms of a limiting value as the value of h approaches zero. i.e.

$$\text{grad}_{f(x)} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{(x+h) - (x-h)}$$

$$\text{grad}_{f(x)} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{2h}$$

1.11 1.12 1.13 2.1 ▸RAD AUTO REAL

CAS Extension

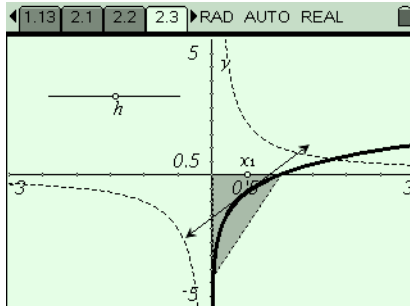
Now consider this process algebraically using computer algebra (if available)!

Enter a function of your choice in cell B1 on the next page and study the steps which follow (you may need to increase column width for column B for some functions).

Then view the graphs of your function and its "derived function"

1.12 1.13 2.1 2.2 ▸RAD AUTO REAL

A	B	C
1	y =	ln(x)
2	P(x, y)	x
3	Q(x, y)	x+h
4	R(x, y)	x-h
5	GradQR	-(ln(x-h)-ln(x...))
B2		x1:=x



2.1 2.2 2.3 3.1 ▸RAD AUTO REAL

Question

6. Use the Lists & Spreadsheet page as a template to show your step-by-step solutions for the functions $3x^2$, $\sin(x)$, \sqrt{x} and $\ln(x)$.

Check your solutions using the program "FirstPrinciples(function)" on the next page.

2.2 2.3 3.1 3.2 ▸RAD AUTO REAL

$R(x_1-h, f(x_1-h)) = \{x_1-h, \ln(-(h-x_1))\}$

The gradient QR is given by $\Delta y / \Delta x$ where

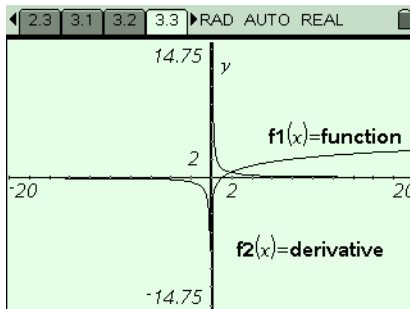
$\Delta y = f(x_1+h) - f(x_1-h)$ and

$\Delta x = (x_1+h) - (x_1-h) = 2h$

i.e. GradientPQ = $\frac{\ln(-(h-x_1)) - \ln(h+x_1)}{2 \cdot h}$

As the value of h approaches zero,

1/1



3.1 3.2 3.3 3.4 ▸RAD AUTO REAL

Question

7. Can you find any functions for which this process does NOT work?

Answer

Functions which are not continuous are not differentiable (such as $\ln(x)$ for $x \leq 0$) and functions which are not smooth are not