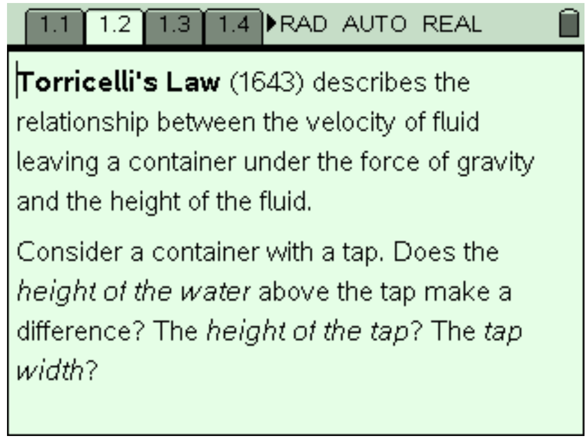
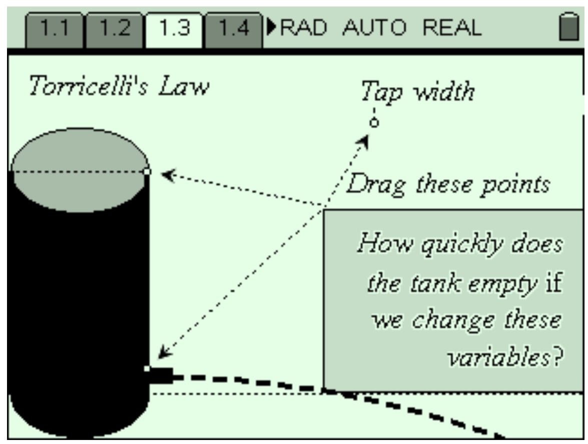
Applications of Calculus: Torricelli's Law ID: XXXX

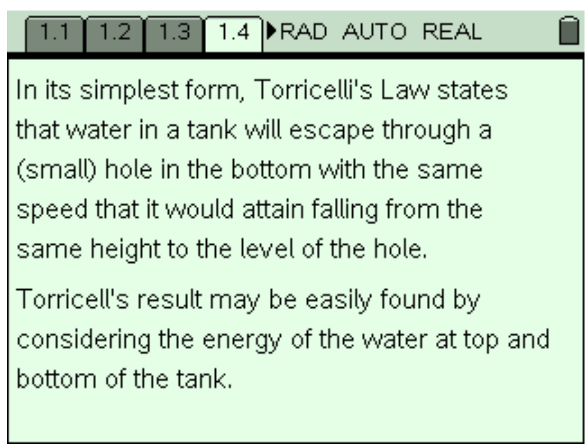
Name ______________________

Class ______________________

In this activity, we explore the application of differential equations to the real world as applied to the velocity at which water escapes from a container.







Open the file *CalcActXX_Torricellis_Law_EN.tns* on your handheld and follow along with your teacher to work through the activity. Use this document as a reference and to record your answers.

EXERCISES

1. Water at the top of the tank has Potential Energy. Give an expression for this in terms of the **mass** of water (**m**), the force of **gravity** (**g**) and the **height** above the tap (**h_1**).

2. Water leaving the tank through the tap has Kinetic Energy. Give an expression for this in terms of the **mass** of the water (**m**) and the **velocity** (**v**) at which it

3. Equate these two expressions (why?) and solve to find the exit velocity of the water as it leaves the tap.

4. Given $v = \sqrt{2 \cdot g \cdot h}$ and $\frac{dV}{dt} = -a_2 v$ (where a_2 is the area of the tap opening and h the height of the water above the center of the tap), use the chain rule to find an expression for $\frac{dV}{dt}$ in terms of $\frac{dh}{dt}$.

5. Simplify this expression using the relationship between the **volume** and **height** of a cylinder.

6. Put these together to find an expression for $\frac{dh}{dt}$ in terms of ***v***, the exit velocity of the water.

7. But from Torricelli we have an expression for ***v*** in terms of ***h***. Put these together!

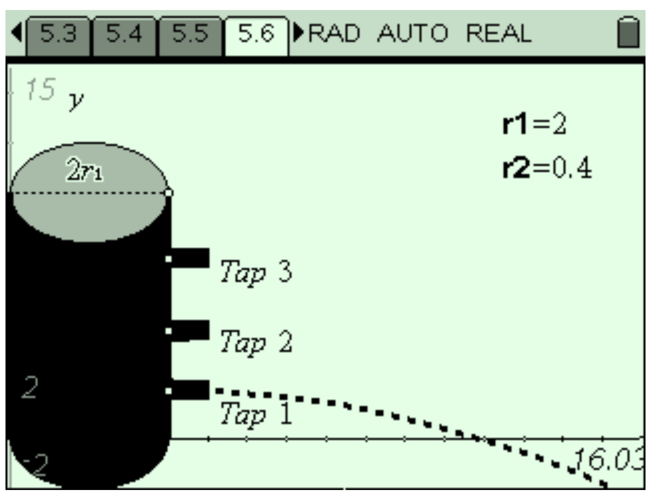
8. Now invert and integrate with respect to ***h*** to derive an expression for ***h*** in terms of ***t***.

9. Interpret this result in terms of our physical situation.

EXTENSIONS

Use principles of projectile motion to derive the trajectory equation for the flow of water.

- Ex1 State the initial conditions.
- Ex2 Derive the horizontal equations of motion.
- Ex3 Derive the vertical equations of motion.
- Ex4 Combine horizontal and vertical components to form the trajectory equation.
- Ex5 Using the model provided, by measurement and calculation, find the trajectory equations for the flow of water from taps 1, 2 and 3.



You may like to check your working and your results for these trajectories using the prepared program, **projectile(initial_x, initial_y, initial_velocity, initial_angle, gravity_constant)** and then type **projectile_fn**.

For example, the path for tap 1 should be **projectile(4, 2,** $\sqrt{2 \cdot 9.8 \cdot 8}$ **, 0, 9.8).**

SUGGESTED SOLUTIONS

1. **PE = m*g*h** (where ***h*** represents the height of the water above the tap centre).

2. **KE =** $\frac{1}{2} \cdot m \cdot v^2$

3. From the law of conservation of energy, PE = KE for a closed system, hence $m \cdot g \cdot h = \frac{1}{2} \cdot m \cdot v^2$ >> $v = \sqrt{2 \cdot g \cdot h}$

4. $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$

5. Since $V = \pi \cdot r^2 \cdot h$ >> $\frac{dV}{dh} = \pi \cdot r^2$ = a_1, the area of the cross-section of the container.

6. $\frac{dV}{dt} = a_1 \cdot \frac{dh}{dt} = -a_2 \cdot v$ **>>** $\frac{dh}{dt} = \frac{-a_2}{a_1} \cdot v = \frac{-{r_2}^2}{{r_1}^2} \cdot v$

7. $\frac{dh}{dt} = \frac{-{r_2}^2}{{r_1}^2} \cdot \sqrt{2 \cdot g \cdot h}$

8. $\frac{dt}{dh} = \frac{-\alpha}{\sqrt{h}}$ **where** $\alpha = \frac{{r_1}^2}{{r_2}^2 \cdot \sqrt{2g}}$ **>>** $t = -2\alpha \cdot \sqrt{h} + C$

   When t = 0, h = h_1 thus $C = -2\alpha \cdot \sqrt{h_1}$ and $t = 2\alpha \cdot (\sqrt{h_1} - \sqrt{h})$

   Then $h = \frac{(\sqrt{h_1} - t)^2}{4\alpha^2}$

9. The relationship between time in seconds and the height of the fluid in the cylinder is quadratic, dependent upon the following parameters:
   a. Height of fluid above the (bottom of the) tap;
   b. Radius of the container;
   c. Radius of the tap hole;
   d. Force of gravity.

   It appears that the fluid drains quickly at first, and then more slowly as it decreases in height. For our situation, the container empties in just under 3 seconds.

Extension

Ex1 When ***t*** = 0:

Displacement: ***x* = 2*r*₁, *y* = *h*₂** *(where r1 is the radius of the cylinder and h2 the height of the tap above the ground)*

Velocity: ***vx* =** $\sqrt{2gh_1}$ and ***vy* = 0**

Acceleration: ***ax*** = 0 and ***ay*** = ***-g***.

Ex2 Horizontal components:

Acceleration: $ax = \frac{d^2}{dt^2}(x) = 0$

Velocity: $vx = \int ax \cdot dt = C1 = \sqrt{2gh_1}$

Displacement: $x = \int vx \cdot dt = t \cdot \sqrt{2gh_1} + C2$

But when t = 0, x = C2 = $2r_1$ and so

$x = 2r_1 + t \cdot \sqrt{2gh_1}$ >> $t = \frac{x - 2r_1}{\sqrt{2gh_1}}$

Ex3 Vertical components:

Acceleration: $ay = \frac{d^2}{dt^2}(y) = -g$

Velocity: $vy = \int ay \cdot dt = C3 - g \cdot t = 0$ when t = 0, hence C3 = 0.

$vy = -g \cdot t$

Displacement: $y = \int vy \cdot dt = C4 - \frac{1}{2} g \cdot t^2 = h_2$ when t = 0 hence C4 = h_2

$y = h_2 - \frac{1}{2} g \cdot t^2$

Ex4 Eliminating ***t*** gives $y = h_2 - \frac{1}{2} g \cdot \frac{(x - 2r_1)^2}{2gh_1} = h_2 - \frac{(x - 2r_1)^2}{4h_1}$

Ex5 Results will depend upon placement of the three taps. For the given position of tap 1 at (2, 4) and gravity at 9.8 m/s2 then **projectile(4,2,** $\sqrt{2 \cdot 9.8 \cdot 8}$ **, 0, 9.8)** produces the quadratic $y = -0.03125 \cdot x \quad + 0.25x + 1.5$