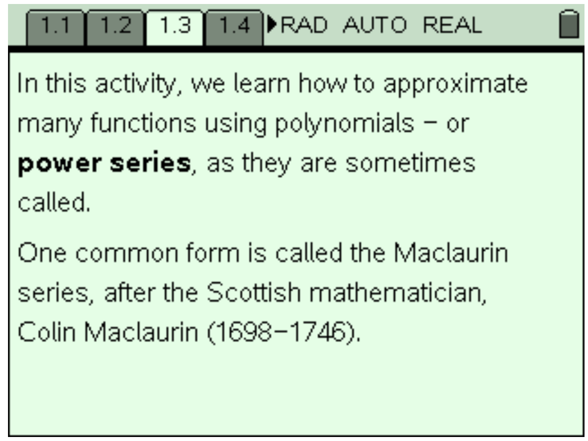
Applications of Calculus: Introducing Maclaurin Series ID: XXXX

Name ______________________

Class ______________________

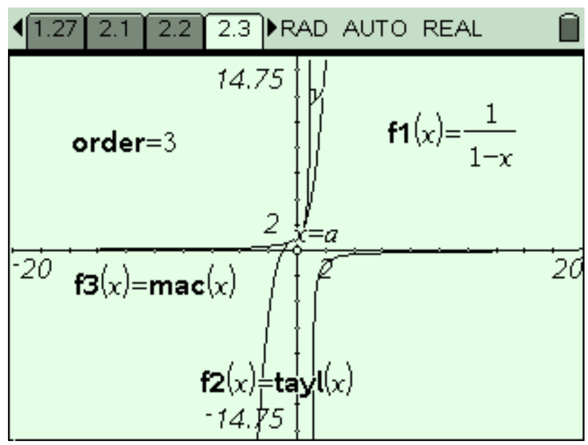
In this activity, we explore the application of calculus to the approximation of functions using power series.

Open the file *CalcActXX_Maclaurin_EN.tns* on your handheld and follow along with your teacher to work through the activity. Use this document as a reference and to record your answers.



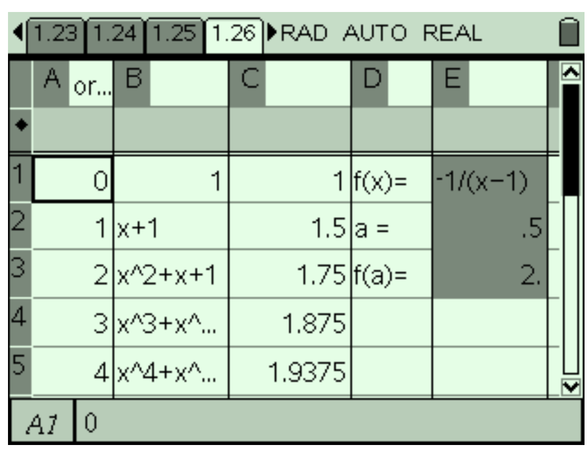
EXERCISES

1. What polynomials of degree zero gives the closest approximation to a curve ***y = f(x)*** near a point ***A (0, f(0))***?



2. What polynomial of degree one gives the closest approximation to ***y = f(x)*** near point ***A (0, f(0))***?

3. Suppose our approximation is of the form $f_n(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n$. By considering the derivatives of order 1, 2, 3, …n and taking ***x = 0***, show that the coefficients for the best polynomial approximation should be: $a_0 = f(0), a_1 = f'(0), a_2 = f''(0), \ldots a_n = f^{(n)}(0)$



4. Hence give the result for the approximating polynomial, $f_n(x)$.

5. Apply the Maclaurin series to the function $f(x) = e^x$ near ***x = 0***.

6. Instead of considering a point A ***(0, f(0))*** with ***x = 0***, consider a more general point with ***x = a***. Substitute ***x – a*** for ***x*** in the Maclaurin series to derive the more general result, called the Taylor series.

7. Show that the Taylor series expansion for $x^2 - 2x - 3$ of order 1 at ***x = 2*** gives the tangent line to the curve at this point.

8. Evaluate the Maclaurin series for the exponential function, e^x, for orders 1, 2, 3, 4 and 5.

9. Clearly describe the relationship between the order of the approximating function and the match with the original function.

10. Use the graph provided to consider the difference between the original function and the Maclaurin series approximation for this function. Comment.

11. Study the spreadsheet provided. Cell E3 shows the value of the function ***f(x)*** at a point ***x = a***, while column B shows the value of the Maclaurin series for the function for increasing orders. What do you observe for differing values of ***a***?

12. You may also use the Taylor series formula on the spreadsheet provided. How does this compare with the Maclaurin series?

13. Finally, study the graphs provided. Identify the Maclaurin series (**mac(x)**) and the Taylor series (**tayl(x)**) curves. What do you observe as you change the value of the point **a** on the x-axis?

EXTENSIONS

- Ex1 Express the Maclaurin series for a function ***f(x)*** of order ***k*** using sigma notation.
- Ex2 Use this formula to evaluate the first five terms of the Maclaurin series for the functions ***sin(x), cos(x), e^x^*** and $\dfrac{1}{1-x}$.
- Ex3 Leonhard Euler showed that $e^{i*x} = \cos(x) + i*\sin(x)$. Prove this remarkable result using Maclaurin series.
- Ex4 Hence prove de Moivre's Theorem, $(\cos(x) + i*\sin(x))^n = \cos(nx) + i\sin(nx)$

SUGGESTED SOLUTIONS

1. The value of the function ***y = f(x)*** at the point ***A (0, f(0))*** is clearly ***f(0)***, so the degree zero "polynomial" which best approximates the function is $a_0 = f(0)$.

2. The tangent at A gives the best degree one fit for this function, ***y = f'(0)*x+f(0)***, hence $a_1 = f'(0)$.

3. If $f^{(n)}_n(x) = n! * a_n$ then $a_n = [f^{(n)}_n(x)]/n!$ Dividing through by ***n!*** gives the required form.

4. $f_n(x) = f(0) + f'(0) * x + f''(0)\dfrac{x^2}{2!} + f^{(3)}(0)\dfrac{x^3}{3!} + ... + f^{(n)}(0)\dfrac{x^n}{n!}$

5. For the function ***f(x) = ex***, $f(0) = f'(0) = f''(0) = \ldots = f^{(n)}(0) = 1$ hence the Maclaurin series expansion is $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + ... + \dfrac{x^n}{n!}$

6. The Taylor series:
$$f_n(x) = f(a) + f'(a) * (x-a) + f''(a)\frac{(x-a)^2}{2!} + f^{(3)}(a)\frac{(x-a)^3}{3!} + ... + f^{(n)}(a)\frac{(x-a)^n}{n!}$$

7. Taylor(x^2 - 2x - 3, x, 1, 2) = 2(x - 2) – 3 which is the tangent.

8. Maclaurin(e^x, 0) = 1
Maclaurin(e^x, 1) = $1 + x$
Maclaurin(e^x, 2) = $1 + x + \dfrac{x^2}{2}$
Maclaurin(e^x, 3) = $1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6}$
Maclaurin(e^x, 4) = $1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24}$
Maclaurin(e^x, 5) = $1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24} + \dfrac{x^5}{120}$

9. As the order increases the approximation curve becomes a better and better fit for the original function at points further and further from zero.

10. As the order increases, the difference function assumes values very close to zero in the interval around the origin. As the order increases, so does the range of this match as the fit improves.

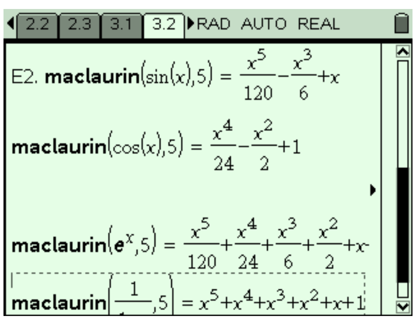
11. As the order increases (columnA), the value of the Maclaurin series approaches the value of the function at ***x = a***. As the value of ***a*** varies from 0 it takes higher orders for it to approach this value.

12. The Maclaurin series is most accurate only around ***a = 0***, while the Taylor series is accurate for any values of ***a***.

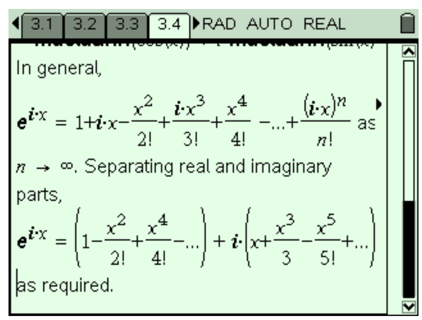
13. The Maclaurin series is unaffected by the value of ***a*** while the Taylor series follows this value, keeping a good approximation centered at ***x = a***.

Extension

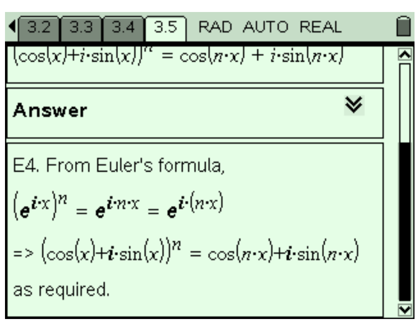
Ex1 Maclaurin(***f(x), k***) = $\sum_{n=0}^{k}\left(\frac{d^n}{dx^n}(f(a))\right)\cdot\frac{x^n}{n!}$



Ex2



Ex3



Ex4