Applications of Calculus: Introducing Maclaurin Series

ID: XXXX

Time required
45 minutes

Activity Overview

In this activity, students are introduced to power series approximations for functions. Using multiple representations to support understanding, students build the foundations for further applications of series, and apply this to the study of Maclaurin and Taylor series. This activity provides an introduction to the applications of differential calculus to series and approximation, and offers consolidation and practice opportunities using dynamic models, algebraic spreadsheets and prepared programs.

Concepts

- *Differential calculus, power series approximations, Maclaurin and Taylor series.*

Teacher Preparation

This investigation offers opportunities for students to begin their studies of power series approximations, deriving Maclaurin and Taylor series expansions for common functions. It offers opportunities for students to consolidate their skills (including using prepared interactive graphing and algebraic spreadsheets). As such, care should be taken to provide ample time for ALL students to engage actively with the requirements of the task, allowing some who may have missed aspects of earlier work the opportunity to build new and deeper understanding.

- *This activity can serve to consolidate earlier work on differentiation, series and approximation. It offers a suitable introduction to power series.*
- *Begin by reviewing calculus applications to approximation methods such as Newton's Method.*
- *The screenshots on pages X–X (top) demonstrate expected student results. Refer to the screenshots on page X (bottom) for a preview of the student .tns file.*
- ***To download the .tns file, go to http://education.ti.com/exchange and enter "XXXX" in the search box.***

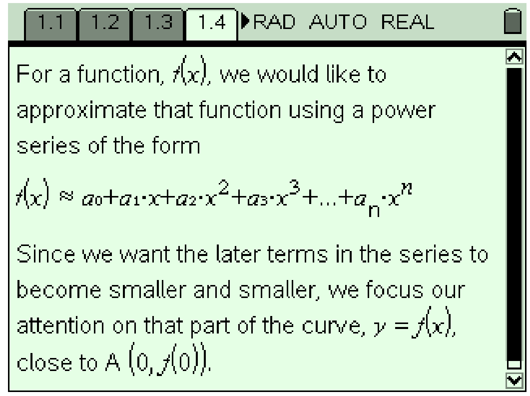
Classroom Management

- *This activity is intended to be **teacher led.** You should seat your students in pairs so they can work cooperatively on their handhelds. Use the following pages to present the material to the class and encourage discussion. Students will follow along using their handhelds, although the majority of the ideas and concepts are only presented in **this** document; be sure to cover all the material necessary for students' total comprehension.*
- *Students can either record their answers on the handheld or you may wish to have the class record their answers on a separate sheet of paper.*

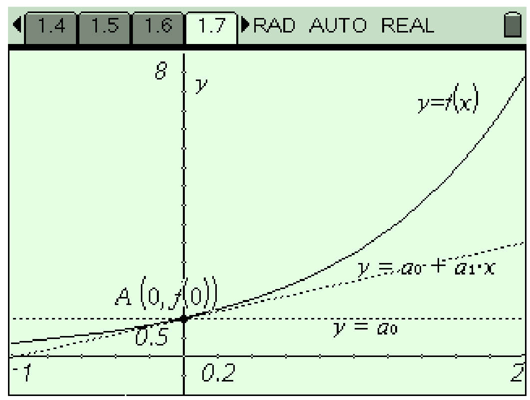
TI-Nspire™ Applications

Notes, Graphs & Geometry, Calculator, Lists & Spreadsheet.

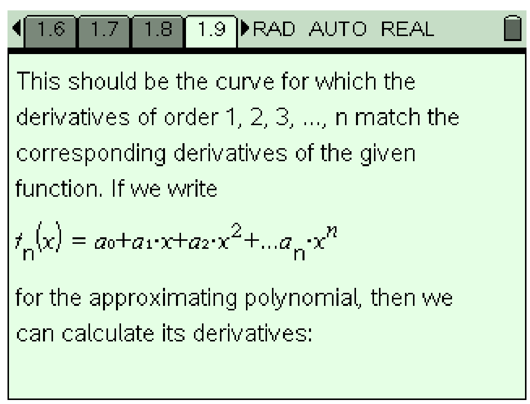
Step 1: The activity begins with discussion of the benefits of polynomials over other, more difficult functions, especially as applied to the calculus. Discussion should center on the practicality of approximating functions using polynomials of various degrees. The Maclaurin series is introduced in this context, as centered around the value ***x = 0***.



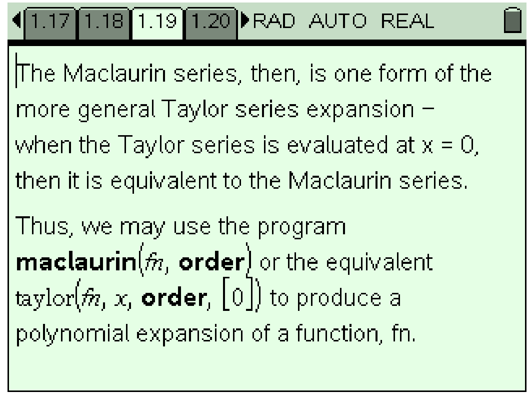
Step 2: Drawing on previous knowledge, students should be encouraged to discuss the degree 0 and 1 cases – the y-intercept of any function offers a good approximation close to ***x = 0***, and the tangent to the curve is likely to offer an even better fit to a given curve.



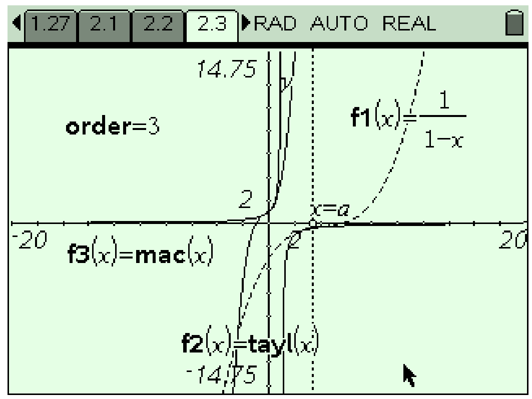
Step 3: Extending these ideas to polynomials of higher degrees lies at the heart of this topic, and teachers need to ensure that students are comfortable with this concept. Further, the relationship between higher derivatives and the “fit” to a curve may need some carefult consideration.



Step 4: Deriving the form for the Maclaurin (and Taylor) series is not difficult and students should be encouraged to attempt this themselves, if necessary with some CAS assistance. It is important that time be spent distinguishing between the Maclaurin and Taylor series, as well as applying these to common functions.

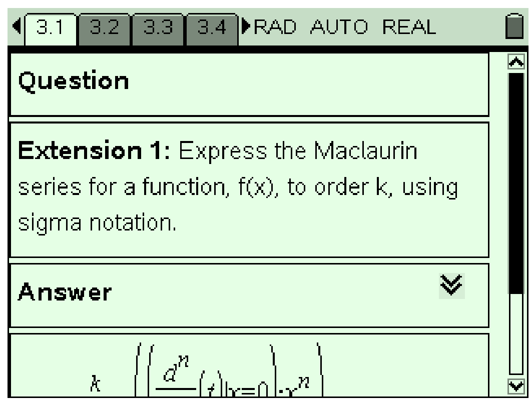


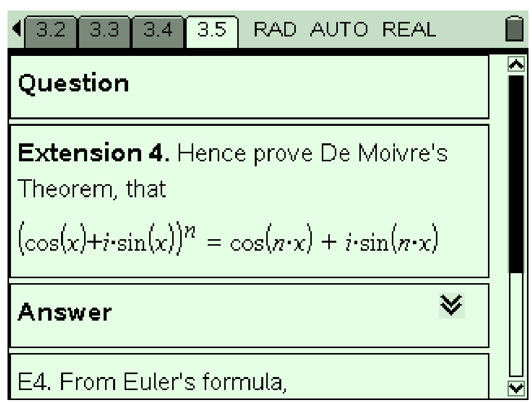
Step 5: Opportunities are provided for students to experiment with the effects of changes to the **order** of the approximation polynomial, and of changes to the center of the approximation (0 for Maclaurin, “a” for Taylor). Interactive graphs and spreadsheets are made available for students to explore these changes and so to be able to express these concepts in their own words.



Extension

The extension activities provide opportunities for students to apply what they have learned, formalizing their derivation of the series using sigma notation, and applying this to further functions. The application is extended to derivation of both Euler’s formula for complex numbers and, from this, de Moivre’s Theorem. Both may be readily established using power series expansions.





Applications of Calculus: Introducing Maclaurin Series – ID: XXXX (Student)TI-Nspire File: *CalcActXX_Maclaurin_EN.tns*

1.1 | 1.2 | 1.3 | 1.4 ▸ RAD AUTO REAL

Introducing Maclaurin Series

Calculus and Series

1.1 | 1.2 | 1.3 | 1.4 ▸ RAD AUTO REAL

Polynomials are great, aren't they? They make it easy to differentiate and integrate and to do all the stuff we like to do for calculus.

If only all functions were as well-behaved as polynomials, then our life would be much simpler.

1.1 | 1.2 | 1.3 | 1.4 ▸ RAD AUTO REAL

In this activity, we learn how to approximate many functions using polynomials – or **power series**, as they are sometimes called.

One common form is called the Maclaurin series, after the Scottish mathematician, Colin Maclaurin (1698–1746).

1.1 | 1.2 | 1.3 | 1.4 ▸ RAD AUTO REAL

For a function, $f(x)$, we would like to approximate that function using a power series of the form

$f(x) \approx a_0+a_1\cdot x+a_2\cdot x^2+a_3\cdot x^3+...+a_n\cdot x^n$

Since we want the later terms in the series to become smaller and smaller, we focus our attention on that part of the curve, $y=f(x)$, close to A $(0,f(0))$.

1.2 | 1.3 | 1.4 | 1.5 ▸ RAD AUTO REAL

Question

1. What polynomial of degree zero gives the closest approximation to the given curve near point A.

Answer

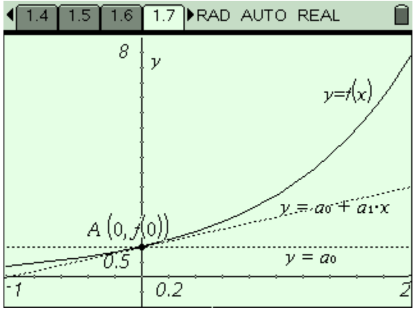
1.3 | 1.4 | 1.5 | 1.6 ▸ RAD AUTO REAL

Question

2. What polynomial of degree one gives the closest approximation to $y=f(x)$ near point A $(0,f(0))$.

Answer

1.4 | 1.5 | 1.6 | 1.7 ▸ RAD AUTO REAL



1.5 | 1.6 | 1.7 | 1.8 ▸ RAD AUTO REAL

In general, we are interested in the polynomial

$y = a_0 + a_1\cdot x + a_2\cdot x^2 + ... + a_n\cdot x^n$

which comes closest to fitting the given curve near A.

This will be the polynomial which has the highest degree of contact with the curve around the given point.

1.6 | 1.7 | 1.8 | 1.9 ▸ RAD AUTO REAL

This should be the curve for which the derivatives of order 1, 2, 3, ..., n match the corresponding derivatives of the given function. If we write

$f_n(x) = a_0+a_1\cdot x+a_2\cdot x^2+...a_n\cdot x^n$

for the approximating polynomial, then we can calculate its derivatives:

1.7 | 1.8 | 1.9 | 1.10 ▸ RAD AUTO REAL

$f'_n(x) = a_1 + a_2\cdot x + a_3\cdot x^2 + ... + a_n\cdot x^{n-1}$

$f''_n(x) = a_2 + a_3\cdot x + a_4\cdot x^2 + ... + a_n\cdot x^{n-2}$

...

$f^{(n)}_n(x) = n!\cdot a_n$

[Check these steps!]

1.8 | 1.9 | 1.10 | 1.11 ▸ RAD AUTO REAL

Question

3. By setting the value of $x = 0$ in the previous equations, show that the coefficients for the best polynomial should be $a_0 = f(0)$, $a_1=f'(0)$, $a_2=\frac{f''(0)}{2!}$...

Answer

1.9 | 1.10 | 1.11 | 1.12 ▸ RAD AUTO REAL

Question

4. Give the result for the approximating polynomial, $f_n(x)$.

Answer

1.10 1.11 1.12 1.13 RAD AUTO REAL

This result gives the Maclaurin series for a given function, $f(x)$.

$$f(x) \approx f(0) + f'(0)\cdot x + \frac{f''(0)\cdot x^2}{2!} + \frac{f^{(3)}(0)\cdot x^3}{3!} \ldots$$

1.11 1.12 1.13 1.14 RAD AUTO REAL

Question

5. Apply the Maclaurin series to the function, $f(x) = e^x$ near $x = 0$.

Answer

1.12 1.13 1.14 1.15 RAD AUTO REAL

We now need to consider whether our approximating polynomial converges to a limiting value of $f(x)$ as $n \to \infty$, i.e.

$$f(x) = \lim_{n\to\infty} \left(f(0) + f'(0)\cdot x + \frac{f''(0)\cdot x^2}{2!} + \frac{f^{(3)}(\ldots}{\ldots}\right.$$

How far, too, may we wander from zero and still have such a result apply?

1.13 1.14 1.15 1.16 RAD AUTO REAL

Question

6. Instead of considering a point, $A\ (0, f(0))$ with $x = 0$, consider a more general point centered at $x = a$. Substitute $x - a$ for x in the Maclaurin series for the more general result.

Answer

1.14 1.15 1.16 1.17 RAD AUTO REAL

This more general result is called the **Taylor series** (after English mathematician, Brook Taylor, 1685 – 1731). This expansion will only apply for functions possessing finite derivatives of all orders at $x = a$.

To evaluate the Taylor series for a function, use **Taylor(fn,x,order,point).**

1.15 1.16 1.17 1.18 RAD AUTO REAL

Question

7. Show that the Taylor expansion for the function x^2-2x-3 of order 1 at $x = 2$ gives the tangent line at this point.

Answer

1.16 1.17 1.18 1.19 RAD AUTO REAL

The Maclaurin series, then, is one form of the more general Taylor series expansion – when the Taylor series is evaluated at x = 0, then it is equivalent to the Maclaurin series.

Thus, we may use the program **maclaurin**(*fn*, **order**) or the equivalent taylor(*fn*, *x*, **order**, [0]) to produce a polynomial expansion of a function, fn.

1.17 1.18 1.19 1.20 RAD AUTO REAL

Question

8. Evaluate the Maclaurin series for the exponential function, e^x, for orders 1, 2, 3, 4 and 5.

Answer

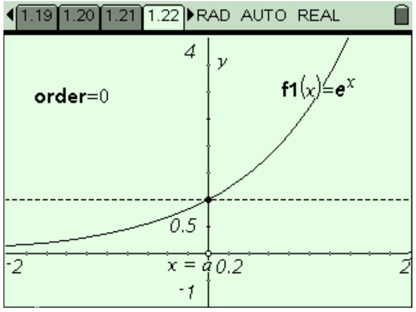
1.18 1.19 1.20 1.21 RAD AUTO REAL

On the Graphs & Geometry page following, the graph of the exponential function is shown, along with its maclaurin series for order 0.

By increasing the value of the variable order, observe the effect upon the approximation function.

Drag the axis point $x = a$ and observe the error between the curves.

1.19 1.20 1.21 1.22 RAD AUTO REAL



1.20 1.21 1.22 1.23 RAD AUTO REAL

Question

9. Clearly describe the relationship between the order of the approximation function and the match with the original function.

Answer

1.21 1.22 1.23 1.24 RAD AUTO REAL

Question

10. Go back to the graph and consider **f3(x)**, the difference between the original function and the maclaurin series approximation. Comment.

Answer

1.22 1.23 1.24 1.25 RAD AUTO REAL

Question

11. Study the spreadsheet on the next page. Cell E3 shows the value of the function ($f(x)$) at a point $x = a$, while column B shows the value for the Maclaurin series at that point. What do you observe for different values of a?

Answer

1.23 1.24 1.25 1.26 RAD AUTO REAL

	A or...	B	C	D	E
1	0	1	1	f(x)=	e^x
2	1	x+1	1.	a =	0
3	2	x^2/2+x...	1.	f(a)=	1.
4	3	x^3/6+x...	1.		
5	4	x^4/24+...	1.		

A1 0

1.24 1.25 1.26 1.27 RAD AUTO REAL

Question

12. You may also use the Taylor series formula in the spreadsheet on the next page.

How does this compare with the Maclaurin series?

Answer

ERROR: undefined
OFFENDING COMMAND: ‘~

STACK: