

## Introducing the Differential Calculus

### Composite Functions

**Time required**

45 minutes

ID: XXXX

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#### Activity Overview

*In this activity, we explore ways to differentiate some more difficult functions. The focus here is on functions which can be expressed as a composition of two functions. The approach taken here is largely symbolic and makes full use of the computer algebra facilities of TI-Nspire CAS. Prepared algebraic spreadsheets are utilized for skill development and consolidation.*

#### Concepts

- *Differentiation of standard forms, fundamentals of derivatives, chain rule.*

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#### Teacher Preparation

*This investigation offers opportunities for review and consolidation of key concepts related to the methods of first principles and differentiation of simple functions. Opportunities are provided for skill development and practice of the method of taking derivatives of composite functions. As such, care should be taken to provide ample time for ALL students to engage actively with the requirements of the task, allowing some who may have missed aspects of earlier work the opportunity to build new and deeper understanding.*

- *This activity can serve to consolidate earlier work on differentiation. It offers a suitable introduction to derivatives of more difficult functions.*
- *Begin by reviewing the method of differentiation of first principles, both graphically and symbolically, and methods of differentiation of the standard function forms.*
- *The screenshots on pages X–X (top) demonstrate expected student results. Refer to the screenshots on page X (bottom) for a preview of the student .tns file.*
- *To download the .tns file, go to <http://education.ti.com/exchange> and enter "XXXX" in the search box.*

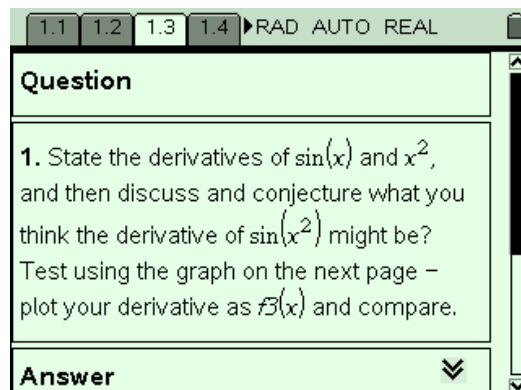
#### Classroom Management

- *This activity is intended to be **teacher led**. You should seat your students in pairs so they can work cooperatively on their handhelds. Use the following pages to present the material to the class and encourage discussion. Students will follow along using their handhelds, although the majority of the ideas and concepts are only presented in **this** document; be sure to cover all the material necessary for students' total comprehension.*
- *Students can either record their answers on the handheld or you may wish to have the class record their answers on a separate sheet of paper.*

#### TI-Nspire™ Applications

*Calculator, Notes, Lists & Spreadsheet and Graphs & Geometry.*

**Step 1:** Begin with discussion and review of both first principles methods and of the derivatives of standard function forms. Ensure that students are comfortable with these and then challenge them to consider more difficult forms – in this case, composite functions of the form  $y = f(g(x))$ .



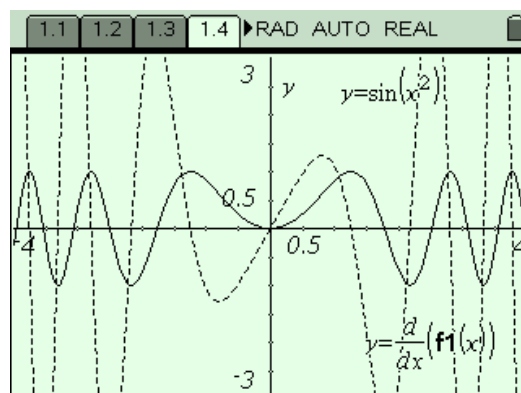
1.1 1.2 1.3 1.4 ▸ RAD AUTO REAL

**Question**

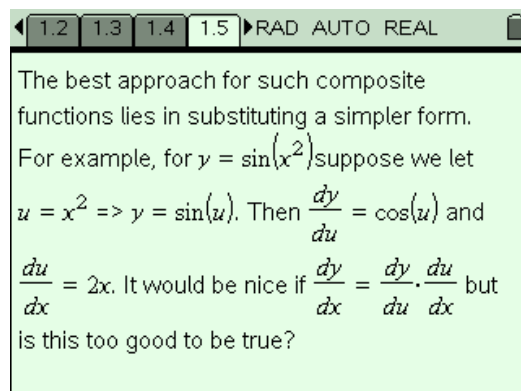
1. State the derivatives of  $\sin(x)$  and  $x^2$ , and then discuss and conjecture what you think the derivative of  $\sin(x^2)$  might be? Test using the graph on the next page – plot your derivative as  $f3(x)$  and compare.

**Answer** ▾

**Step 2:** A graphical approach is quite appropriate here as a means by which students may make and test conjectures – the device shows the function and the graph of its derivative – students try their own ideas and readily see whether these are correct.



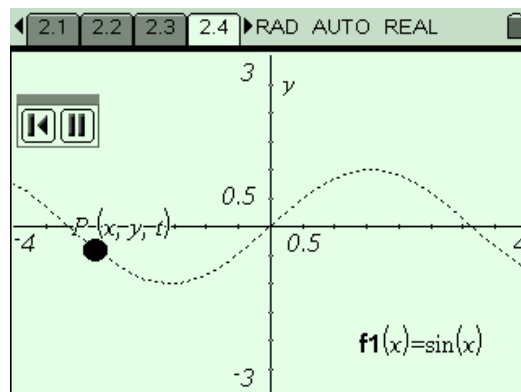
**Step 3:** Students will probably have little trouble coming up with the chain rule, nor accepting that it is valid, but they should be encouraged to question and check through the justification of this method – for this they need to go back to fundamental ideas of limits and whether these can be treated “like fractions”.



1.2 1.3 1.4 1.5 ▸ RAD AUTO REAL

The best approach for such composite functions lies in substituting a simpler form. For example, for  $y = \sin(x^2)$  suppose we let  $u = x^2 \Rightarrow y = \sin(u)$ . Then  $\frac{dy}{du} = \cos(u)$  and  $\frac{du}{dx} = 2x$ . It would be nice if  $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$  but is this too good to be true?

**Step 4:** As part of this investigation, the introduction of parametric forms – especially in the sense of adding a “time variable” to the graphing of a function – sets the scene for much later work. It would be well to allow students time for a thorough treatment of this concept. They should also be expected to put these understandings into their own words.



**Step 5:** Having derived the chain rule for composite functions, it remains only for students to take the time to consolidate its application, and worked examples are provided.

2.7 2.8 2.9 2.10 RAD AUTO REAL

**Question**

6. Use this "chain rule" to find the derivative of  $y = \sin(x^2)$ . Check your answer on the next page.

**Answer**

Let  $u = x^2$ , then  $\frac{du}{dx} = 2x$  and  $\frac{dy}{du} = \cos(u)$ .

**Step 6** An algebraic spreadsheet is available to support students in working through the process: students supply each step, which is checked for algebraic equivalence.

2.8 2.9 2.10 2.11 RAD AUTO REAL

A	B	C
1 $y = f(x) \gg$	$\sin(x^2)$	
2 $u =$	$x^2$	
3 $y = f(u) \gg$	$\sin(u)$	✓
4 $dy/du =$	$\cos(u)$	✓
5 $du/dx =$	$2*x$	✓
A1	$"y = f(x) \gg"$	

This scaffolding tool may be used as much or as little as desired, noting that it does offer a model for a well-structured worked solution for such questions.

Finally, after working through several examples, students should be encouraged to apply these ideas to the general case.

2.12 2.13 2.14 2.15 RAD AUTO REAL

**Question**

10. Clearly describe this method for a composite function,  $y = f(g(x))$ .

**Answer**

Let  $u = g(x)$ , then  $y = f(u)$

$\frac{du}{dx} = g'(x)$  and  $\frac{dy}{du} = f'(u)$

Introducing the Differential Calculus: Composite Functions –

ID: XXXX

(Student)TI-Nspire File: CalcActXX\_Composite\_Functions\_EN.tns

1.1 1.2 1.3 1.4 ▶RAD AUTO REAL

**Introducing the Differential Calculus: Composite Functions**

**Calculus with CAS**

1.1 1.2 1.3 1.4 ▶RAD AUTO REAL

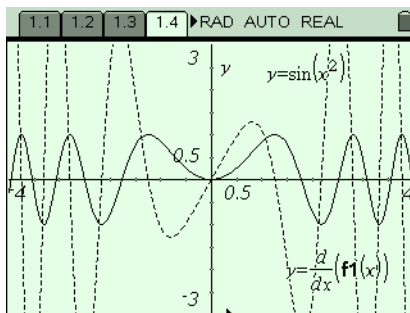
The rules for the differentiation of the standard function forms are readily derived by using methods such as a First Principles approach – or even a graphical approach. More complicated forms, however, require a little more work. In this activity, we consider composite functions, of the form  $f(g(x))$ .

1.1 1.2 1.3 1.4 ▶RAD AUTO REAL

**Question**

1. State the derivatives of  $\sin(x)$  and  $x^2$ , and then discuss and conjecture what you think the derivative of  $\sin(x^2)$  might be? Test using the graph on the next page – plot your derivative as  $F(x)$  and compare.

**Answer**



1.2 1.3 1.4 1.5 ▶RAD AUTO REAL

The best approach for such composite functions lies in substituting a simpler form. For example, for  $y = \sin(x^2)$  suppose we let  $u = x^2 \Rightarrow y = \sin(u)$ . Then  $\frac{dy}{du} = \cos(u)$  and  $\frac{du}{dx} = 2x$ . It would be nice if  $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$  but is this too good to be true?

1.3 1.4 1.5 2.1 ▶RAD AUTO REAL

**Question**

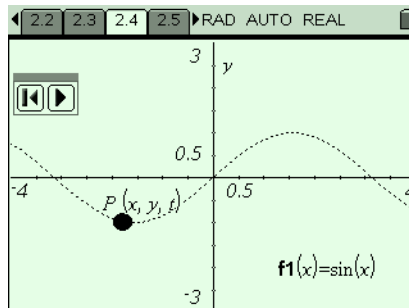
2. Such a result suggests that the derivatives (which are actually limits from first principles work we have done) can behave very much like fractions (at least in some cases!) Is this reasonable?

1.4 1.5 2.1 2.2 ▶RAD AUTO REAL

Expressions such as  $dy/dx$  are called infinitesimals, since they represent a limit as the small value  $dx$  approaches zero. i.e. From first principles approaches, we can easily establish that  $\lim_{\Delta x \rightarrow 0} \left( \frac{\Delta y}{\Delta x} \right) = \frac{dy}{dx}$  where  $\Delta x$  and  $\Delta y$  are very small (non-zero) amounts.

1.5 2.1 2.2 2.3 ▶RAD AUTO REAL

Now introduce another variable, say  $t$ , related to  $x$  and  $y$  –  $t$  may be even be thought of as representing the position of a point  $(x, y)$  on a curve  $y = f(x)$  at a particular time, as the point traces out that curve. This idea becomes the basis for parametric equations. Instead of thinking of the curve  $y = f(x)$  as a static object, just picture it being traced out in animation!



2.2 2.3 2.4 2.5 ▶RAD AUTO REAL

**Question**

3. Carefully explain the relationship between  $x$ ,  $y$  and  $t$  in the preceding graph.

**Answer**

2.3 2.4 2.5 2.6 ▶RAD AUTO REAL

Clearly, in addition to the relationship defined between  $x$  and  $y$  by the function,  $y = f_1(x)$ , there also exist relationships of the form  $y = u(t)$  and  $t = v(x)$  (for some functions  $u$  and  $v$ ). We seek to define the relationship between derivatives  $\frac{dy}{dx}$ ,  $\frac{dy}{dt}$  and  $\frac{dt}{dx}$ .

2.4 2.5 2.6 2.7 ▶RAD AUTO REAL

**Question**

4. Eliminate the variable  $t$  in these equations to define a composite function for  $y$ .

**Answer**

2.5 2.6 2.7 2.8 ▸RAD AUTO REAL

By simple algebra we know that

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta t} \cdot \frac{\Delta t}{\Delta x} \text{ for any real, small values.}$$

Then  $\lim_{\Delta x \rightarrow 0} \left( \frac{\Delta y}{\Delta x} \right) = \lim_{\Delta x \rightarrow 0} \left( \frac{\Delta y}{\Delta t} \cdot \frac{\Delta t}{\Delta x} \right)$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \left( \frac{\Delta y}{\Delta t} \right) \cdot \lim_{\Delta x \rightarrow 0} \left( \frac{\Delta t}{\Delta x} \right)$$

2.6 2.7 2.8 2.9 ▸RAD AUTO REAL

**Question**

5. Since we know that  $x$ ,  $y$  and  $t$  are related, then we can safely assume (for continuous and differentiable functions, at least) that as  $\Delta x \rightarrow 0$  then  $\Delta t \rightarrow 0$ .

Use this to simplify the result.

**Answer**

2.7 2.8 2.9 2.10 ▸RAD AUTO REAL

**Question**

6. Use this "chain rule" to find the derivative of  $y = \sin(x^2)$ . Check your answer on the next page.

**Answer**

2.8 2.9 2.10 2.11 ▸RAD AUTO REAL

	A	B	C
1	$y = f(x) >>$	?	
2	$u =$	?	
3	$y = f(u) >>$	?	✓
4	$dy/du =$	?	X
5	$du/dx =$	?	X
A.1	$"y = f(x) >>"$		

2.9 2.10 2.11 2.12 ▸RAD AUTO REAL

**Question**

7. Now use the same method to find the derivative of  $\sin(x)^2$

**Answer**

2.10 2.11 2.12 2.13 ▸RAD AUTO REAL

**Question**

8.  $\frac{d}{dx}(\ln(x^2))$

**Answer**

2.11 2.12 2.13 2.14 ▸RAD AUTO REAL

**Question**

9.  $\frac{d}{dx}(\sin(\tan(x)))$

**Answer**

2.12 2.13 2.14 2.15 ▸RAD AUTO REAL

**Question**

10. Clearly describe this method for a composite function,  $y = f(g(x))$ .

**Answer**