Introducing Variables: From Pegs to Patterns

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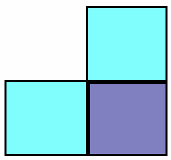
1, 2, 3…

What is the next step in this pattern? Did you say 4? What about 5? Could it be 5? What about 6? It could be any of these and more!



What about the next step in this pattern? Can you see a few possibilities? Pierre van Hiele speaks of mathematics education in terms of *structure* and *insight* (van Hiele, 1986). Structure is the way we make sense of the world: when we see patterns and regularities in everything from bus timetables to music, we are identifying structure. He speaks of a "network of relations".

Insight is a recognition of structure. When we identify a structure and know what to do to continue it, we are able to act appropriately and with intention. Listen to a piece of music for the first time: often, you will decide whether you like it or not quite quickly, and this may well be because you can predict what will come next! You have recognised structure and experienced insight.

Our teaching in schools, especially in mathematics, may be seen to revolve around these two principles. We would like our students to be able to know what to do, to act correctly with purpose and direction in a variety of situations, and we help them in this by leading them to recognise a wide variety of structures. Mathematics has been described as a *search for patterns and relationships*, which is just another way of saying that it is a search for *structure*!

But not all structures are well-defined. Van Hiele spoke of structures as *feeble* or *rigid*. Some patterns can be extended with confidence, whether they be patterns of number, geometrical design, music or art. These may be thought of as *rigid structures*. Others inspire less confidence: we may suspect what comes next, but are unsure, or we see a variety of possible extensions. These structures are *feeble*.

The two patterns given above both appear rigid at first, but this is deceptive. They may be continued in a variety of ways. And this is true for many of the patterns we use in our classrooms! You might argue that any pattern is feeble without sufficient terms to define it, and this is true. Certainly, we need to provide enough terms to be confident of the true nature of the pattern. But in fact, give me any number pattern, with any number of terms, and I can produce a mathematical rule which can allow the next term to be anything you choose!

Think of all those early algebra examination questions of the form "continue this pattern". This is risky business, and those students who insist on seeing things from a slightly different perspective from their teacher and peers are regularly penalised for being creative!

What can we do about this problem that is realistic for real classrooms? In fact, there are two ingredients which need to be added in order to help here: the use of the *concrete*, and the use of *language*. Both are necessary ingredients for good learning, but are particularly appropriate for teaching algebra in a meaningful way. In the past, we have made two fundamental mistakes in our teaching of so much of school mathematics, leading to poor understanding, bad attitudes, and lifelong aversion to our subject. The two key errors have been:

1. We have moved our students too quickly from the concrete to the cognitive.
2. We have failed to make use of students' own language as the essential mediating step in this process.

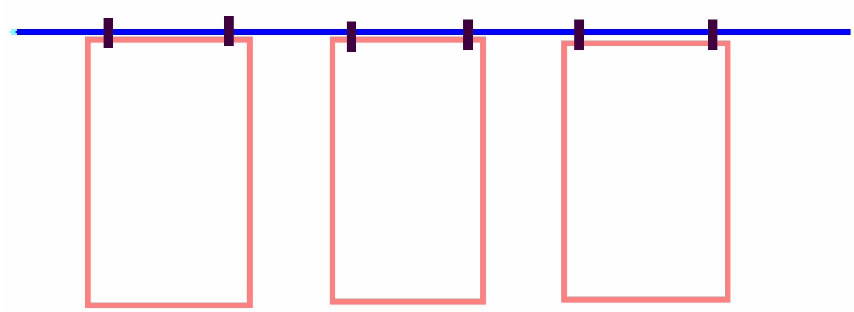
I would suggest that our primary task as teachers lies in assisting our students to move from the concrete (from doing) to the cognitive (to thinking, ideas, understanding), and this is best achieved using language as a mediating step. These ideas are not new, and certainly not my own. They go back to Piaget, to Vygotsky, and to others. Van Hiele identifies five steps in the process of good teaching leading to good learning:

1. *Information*: Students begin to find out about the (cognitive) object or domain of study.
2. *Guided Orientation*: The teacher directs them in activities which allow them to engage (play with) that to be learned.
3. *Explicitation*: Students are required to put their observations into their own words, and share these with peers and teacher.
4. *Free Orientation*: Students are then provided with opportunities to engage more freely, perhaps in the context of an open-ended problem or question, which drives them to explore more deeply.
5. Integration: The level of understanding has moved from being superficial to being deep, articulate and richly connected.

Van Heile's work has often been identified with learning in the field of geometry, but is equally informative across all domains of learning. It is a rich and powerful source of ideas and observations for teachers examining their craft. Here we apply aspects of it to the teaching of early algebra. In particular, to moving from patterns to variables.

Hanging out the Washing

As I went outside this morning to hang some towels on the clothesline, I began thinking about the number of pegs I would need. I hang each towel separately, with a peg at each corner, as shown. I do not have many towels to hang, these days.



Doing this activity in class, I used string, some (clean) handkerchiefs, and pegs brought from home. I asked a student to come and peg up the first "towel", another for the next, and then a third student to peg up a third towel. I then asked them to write a sentence connecting the "*number of towels*" and the "*number of pegs*". This was not difficult, and most readily arrived at statements of the form:

The number of pegs I need = *2* x *the number of towels.*

Students were then asked to complete a table of values for different numbers of towels.

Number of Towels	Number of pegs
1	2
2	4
3	6
4	8

At this stage, of course, my students thought I was crazy, because this was so obvious! But I did point out to them that a rule can be a very powerful thing: it allows me to predict exactly the number of pegs I would need to grab to hang out any number of towels. I pointed out, too, that mathematics is a very economical language and, while we need to *think* and *say* the words, we may use letters to stand for these different things. We could use, say, the letter T for the number of towels, and P for the number of pegs, and write an equation which connects them.

Asked to write such a rule, many (most?) students wrote T = 2 x P, so when T = 1 and P = 2, then this predicts that 1 = 2 x 2 = 4. Oops! Go back to the words and the table: the number of *pegs* is twice the number of *towels.* Thus P = 2 x T. That is better! In fact, we can use any letters we like here, and sometimes it is helpful NOT to use the obvious ones, because this can lead to confusion. My graphing calculator uses *x* and *y* for such rules.

X	Y1	
1	2	
2	4	
3	6	
4	8	
5	10	
6	12	
7	14	

X=1

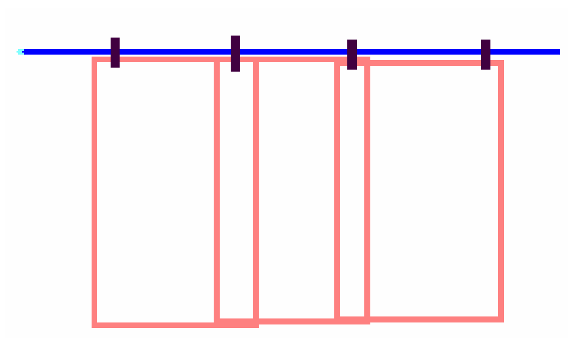
Write down the rule for this table of values. Compare your rule with that of a partner: do you both agree. Discuss and decide which one is correct. Did you agree that $y = 2x$?

Before we move on, we might look at this simple problem in a slightly different way. I asked two students to hang out a new set of towels using a cooperative approach. One student would be in charge of attaching the first peg, the other student would attach the second peg. *How many pegs would each student need to grab for, say, 5 towels?*

Again, this is easy, but it does illustrate that the rule could have been expressed differently. *The number of pegs needed would be the same as the number of towels PLUS the number of towels again!* (P = T + T or $y = x + x$). Part of the power (and challenge) of algebra lies in the different ways which may be used to express the same essential idea, or rule.

Hanging for a Family

A few years ago, we had all our four children living at home, and hanging out towels in the morning was a bigger enterprise. I realised that I used a different method back then. Each towel *overlapped* the next one, sharing a peg between them, as shown. Once again, I first asked one student to come out and hang each towel. Each took two pegs, but only the first student used both!



They were easily able to form the table of values, and when asked to write the rule using words, eventually decided that

The number of pegs should equal the number of towels PLUS one more peg.

Using symbols, P = T + 1 (the challenge, for those willing, produced $y = x + 1$.

X	Y1	
1	2	
2	3	
3	4	
4	5	
5	6	
6	7	
7	8	

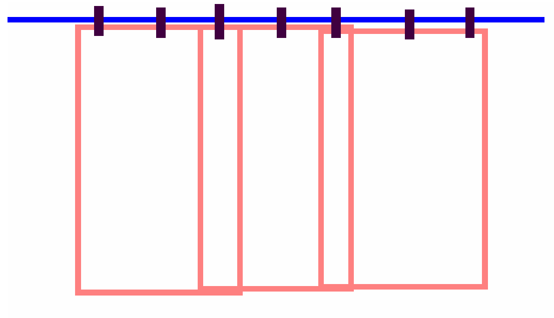
X=1

This was further reinforced by using “cooperative hanging”: two students working together. The first placed the first corner peg on each towel, and the second was left with only one peg left to place!

Mother Knows Best

My mother is very particular about her towels. Two pegs per towel can cause them to become stretched, but she is economical too, so she does overlap her towels. She simply adds a peg in the middle of each towel to keep their shape!

Find the rule for this pattern!



X	Y_1	
1	3	
2	5	
3	7	
4	9	
5	11	
6	13	
7	15	

X=1

Begin with the table of values: how many pegs for 1 towel? For two towels? And so on.

In how many different ways can this rule be written?

Students are encouraged to physically model the problem – to come out and hag towels with others to explore the variations.

If we allocate one person to hang each towel, then each will probably grab 3 pegs, but only the first will use all three – the rest will only need 2. This gives the rule:

The number of pegs I need equals Twice the number of towels PLUS one more.

i.e. $y = 2x + 1$

Cooperative hanging using two people (one for the first corner peg, another for the middle peg) produces the rule $y = x + (x + 1)$.

Three partners sees the first two needing one peg for each towel, and the third needing to add just one more peg: $y = x + x + 1$.

Stress that each rule MUST be put into words in a way that makes sense – compare with a partner. Argue if necessary.

Conclusion

Where you go from here is up to you, but be confident that you have laid solid ground rules for further study of variables. Note the importance of having a concrete referent; a physical model which students can refer to in building their number patterns?

Too often, we move too quickly to abstract number patterns which, as we have seen, can actually be deceiving in terms of what comes next. A physical model ensures that there is a correct "next term" AND that they have the means to find it, and then compare to the number pattern being built. Do not be too quick to move students from the physical to the cognitive – there are many great opportunities in concrete patterns available. And ALWAYS demand that they put their rules and patterns into words that make sense, and which can be compared with others.

Van Hiele, P. (1986). *Structure and Insight: A theory of mathematics education*. Orlando, FA: Academic Press.